Convolution algebras via Chow groups

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"Reconstruction" problems:

1) $\mathcal{A}$ abelian cat. $M \in \mathcal{A}$

$$\mathrm{Hom}(M,-) \quad \mathcal{A} \xrightarrow{\sim} \mathrm{mod}-\mathrm{End}(M)$$

(small coproduct, M cpt proj. generator)

2) $\mathcal{C}$ idempotent complete triang cat.
$\cup$
$\mathcal{J}$ family of objects

$$\bigoplus_{M \in \mathcal{J}} \mathrm{Hom}_{\mathcal{C}}(M,-): \langle \mathcal{J} \rangle_{\cong, \oplus, \mathcal{C}} \xrightarrow{\sim} \mathrm{mod}_{fgp} - \mathrm{End}_{\mathcal{C}}(\mathcal{J})$$

$$\mathrm{End}_{\mathcal{C}}(\mathcal{J}) \overset{!!}{=} \bigoplus_{M,N \in \mathcal{J}} \mathrm{Hom}(M,N)$$

3) $A = \bigoplus_{n \in \mathbb{N}_0} A_n$ positively graded

$A_0 = \bigoplus_j L_j \leftarrow$ simple

Koszul

$$A^! = \mathrm{Ext}_A^\circ(A_0, A_0)$$

$$\mathcal{J} = \langle L_j \langle i \rangle [i] \rangle$$

internal, homol.

$\mathcal{D}^b(A\text{-gmod}) \xrightarrow{\sim} \mathcal{D}^{\downarrow}(A^!\text{-gmod})$ Koszul duality

$\langle \mathcal{J} \rangle_{\cong, A} \xrightarrow{\sim} \mathcal{D}_{perf}(A^!\text{-gmod}).$

4) $\mathcal{A}$ highest weight category

$\cup$

$\mathcal{J}$ isomorphism classes of indec. tilting objects

$E := \operatorname{End}(\mathcal{J})$ $\quad \operatorname{mod}_{cfd}(E)$ Ringel dual category.

$\langle \mathcal{J} \rangle_{\cong, \Delta} \xrightarrow{\sim} \mathcal{D}_{perf}(E)$

Warning: In general $\operatorname{End}(\mathcal{J})$-mod doesn't recover the category (without passing to A_∞-structures)

Geometric repr. theory gives examples

- of Koszul algebras A
- with gradings from homological / geometric origin
- $A = \bigoplus_{i,j} \operatorname{Ext}(\mathcal{L}_i, \mathcal{L}_j)$ $\leftarrow$ simple perverse sheaves

Often:

Formality (Rider, Schnürer, McNamara, ...)

Setup: varieties over $\overline{\mathbb{F}_p}$, coeff. in $\mathbb{Q}$

X_i smooth $(i \in I)$

$\mu_i \downarrow$ proper G-equiv. $\quad G$ affine alg. group

$\mathcal{N}$

$$\rightsquigarrow \boxed{E^{(G)}_{conv} := \bigoplus Ch^G_*(X_i \times_{\mathcal{N}} X_j)} \quad \text{motivic convolution algebra.}$$

$$E_{mot} = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{i,j} \mathrm{Hom}(\mu_{i!} \mathbb{Q}_{X_i}, \mu_{j!} \mathbb{Q}_{X_j}[2n]\langle j \rangle)$$

$$DM^{Spr}_G(\mathcal{N}, \mathbb{Q}) := \langle \mu_{i!} \mathbb{Q}_{X_i} \rangle_{\simeq, \oplus, \in, \Delta} \subseteq ?$$

Theorem: 1) Assume

(PT) $\forall x \in \mathcal{N}$ the $\mu^{-1}(x)$ pure Tate

(FO) $\mu_i(X_i) \subseteq \mathcal{N}$ has finitely many G-orbits.

Then: $DM^{Spr}_G(\mathcal{N}, \mathbb{Q}) \xrightarrow{\sim} \mathcal{D}^{\mathbb{Z}}_{perf}(E_{conv})$

$\cap$

$DM_G(\mathcal{N}, \mathbb{Q})$ equivariant motivic sheaves on $\mathcal{N}$ (Voevodsky)

2) $E_{conv} \cong E_{mot}$ graded algebras

$\langle 1 \rangle$ internal   will shift $(1)[2]$

$\uparrow$ Tate shift

Main tool:

Prop: Assuming (PI), (PO) the

$\mathcal{J}^{Spr}$ = family of objects $\mu_{i!} \mathbb{Q}_{X_i}$

tilting family

(i.e. $\mathrm{Hom}(M, N[n]) = 0 \quad n \neq 0$
$\forall M, N \in \mathcal{J}^{Spr}$).

Convolution

$$\begin{array}{c} X_1, X_2, \ldots \\ \mu_1 \searrow \ \swarrow \mu_2 \ \ldots \\ W \end{array}$$

smooth varieties $/k = \bar{k}$

μ_i proper

W not necessarily smooth

$$\begin{array}{c} X_j \times_W X_k \xleftarrow{pr} \\ \times \\ X_i \times_W X_j \xleftarrow{pr} \end{array} X_i \times_W X_j \times X_j \times_W X_k \xleftarrow{1 \times \Delta \times 1} X_i \times_W X_j \times_W X_k \xrightarrow{pr} X_i \times_W X_k$$

$$(\alpha, \beta) \longmapsto \alpha * \beta := pr_* \circ \Delta^! ((\alpha, \beta))$$

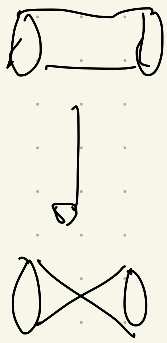
$$\begin{array}{c} Ch(X_j \times_W X_k) \\ \times \\ Ch(X_i \times_W X_j) \end{array} \xrightarrow{\text{convolution}} Ch(X_i \times_W X_j)$$

Chow groups of cocycles / rational equivalence
(work with $\mathbb{Q}$ coefficients)

Examples ?

1)
$$\begin{tikzcd} \tilde{\mathcal{N}} \arrow[d, "\mu"'] \\ \mathcal{N} \end{tikzcd}$$

Springer resolution



$\mathcal{N}$ = nilpotent elements in ss. Lie algebra $\mathfrak{g}$

$$E^0 = Ch^0(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}) = \mathbb{Q}[\text{Weyl group}]$$

2) $Q = (Q_0, Q_1)$ quiver

$$\begin{tikzcd} Q(\underline{d}) := \{\text{flagged representations of dim vector } d \text{ and flag type } \underline{d}\} \arrow[d, "\text{forget flag} = \mu"'] \\ Rep_d \end{tikzcd}$$

$G := \prod_{i \in Q_0} Gl_{d_i}$ - equivariant

$$\boxed{E := \bigoplus_{\underline{d}, \underline{d}'} Ch^G(Q(\underline{d}) \times_{Rep_d} Q(\underline{d}'))}$$

vector compositions of d

Motivic KLR-algebra (Khovanov-Lauda, Rouquier, Varagnolo-Vasserot)

Motivic Quiver Schur algebra (S.-Webster)

3) $X = G/B$ flag variety

$\cup$

$\overline{X}_w$ Schubert variety (w ∈ Weyl group)

$\cup$

X_w Schubert cell

$$\begin{array}{l} BS(\underline{w}) \\ \downarrow \mu_w = \text{Bott-Samelson resolution of } X_w \\ X \end{array}$$

T-equivariant

$\rightsquigarrow E := \bigoplus_{w,w'} Ch\left(BS(\underline{w}) \times_{G/B} BS(\underline{w}')\right)$

endomorphism algebra of certain Soergel bimodules

4) (Graded) Hecke algebras (Lusztig) ...

⋮

Weight structures

Definition A.2. [Bon10, Definition 1.1.1] *Let $\mathcal{C}$ be a triangulated category. A* weight structure $\boldsymbol{\omega}$ *on $\mathcal{C}$ is a pair* $\boldsymbol{\omega} = (\mathcal{C}^{w\leq 0}, \mathcal{C}^{w\geq 0})$ *of full subcategories of $\mathcal{C}$, which are closed under direct summands, such that with $\mathcal{C}^{w\leq n} := \mathcal{C}^{w\leq 0}[-n]$ and $\mathcal{C}^{w\geq n} := \mathcal{C}^{w\geq 0}[-n]$ the following conditions are satisfied:*

(1) $\mathcal{C}^{w\leq 0} \subseteq \mathcal{C}^{w\leq 1}$ *and* $\mathcal{C}^{w\geq 1} \subseteq \mathcal{C}^{w\geq 0}$;

(2) for all $X \in \mathcal{C}^{w\geq 0}$ and $Y \in \mathcal{C}^{w\leq -1}$, *we have* $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$;

(3) for any $X \in \mathcal{C}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \xrightarrow{+1}$$

with $A \in \mathcal{C}^{w\geq 1}$ *and* $B \in \mathcal{C}^{w\leq 0}$.

The full subcategory $\mathcal{C}^{w=0} = \mathcal{C}^{w\leq 0} \cap \mathcal{C}^{w\geq 0}$ *is called the* heart *of the weight struture.*

Definition A.1. [BBD82, Definition 1.3.1] *Let $\mathcal{C}$ be a triangulated category. A* t-structure t *on $\mathcal{C}$ is a pair $t = (\mathcal{C}^{t\leq 0}, \mathcal{C}^{t\geq 0})$ of full subcategories of $\mathcal{C}$ such that with $\mathcal{C}^{t\leq n} := \mathcal{C}^{t\leq 0}[-n]$ and $\mathcal{C}^{t\geq n} := \mathcal{C}^{t\geq 0}[-n]$ the following conditions are satisfied:*

(1) $\mathcal{C}^{t\leq 0} \subseteq \mathcal{C}^{t\leq 1}$ *and* $\mathcal{C}^{t\geq 1} \subseteq \mathcal{C}^{t\geq 0}$;

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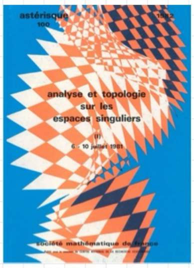
with $A \in \mathcal{C}^{t\leq 0}$ *and* $B \in \mathcal{C}^{t\geq 1}$.

The full subcategory $\mathcal{C}^{t=0} = \mathcal{C}^{t\leq 0} \cap \mathcal{C}^{t\geq 0}$ *is called the* heart *of the t-struture.*

Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)

M.V. Bondarko



also require all categories to be idempotent complete

Ex: 1) Stupid filtration weight structure:

$\mathcal{A}$ idempotent complete additive category

$$K^b(\mathcal{A})^{w\geq 0} = \langle \underset{\sim}{X} \mid X_i = 0 \ \forall i < 0 \rangle / \cong$$

$$K^b(\mathcal{A})^{w\leq 0} = \langle X \mid X_i = 0 \ \forall i > 0 \rangle / \cong$$

$\heartsuit := \mathcal{A}$ (additive) category.

2) $\mathcal{A}$ abelian category, $\operatorname{projdim} \mathcal{A} < \infty$

$$D^b(\mathcal{A}) = K^b(\mathrm{Proj}(\mathcal{A})) \qquad \heartsuit = \mathrm{Proj}(\mathcal{A})$$

General construction (Bondarko)

$\mathfrak{T} \subseteq \mathcal{C}$ triang. cat, idempotent complete

Collection of objects

Assume: • $\mathfrak{T}$ negative i.e. $\mathrm{Hom}(X, Y[n]) = 0 \quad n > 0 \quad \forall X, Y \in \mathfrak{T}$

• $\langle \mathfrak{T} \rangle_\Delta = \mathcal{C}$

$\Rightarrow \exists$ weight structure with $\heartsuit = \mathcal{C}^{\omega=0} = \langle \mathfrak{T} \rangle_{\cong, \oplus, \ominus}$

Beilinson's realisation functor: $\boxed{D^b(\mathcal{C}^{t=0}) \longrightarrow \mathcal{C}}$

t-structure

$\mathcal{C}^{t=0} = \heartsuit$

Bondarko's weight complex functor:

$$\boxed{\mathrm{wt} : \mathcal{C} \longrightarrow K^b(\mathcal{C}^{\omega=0})}$$

$\mathcal{C}^{\omega=0} = \heartsuit$

⊗ Case: bounded weight structure, $\mathcal{C} = h\,\mathcal{C}_\infty$)

Prop: Assume ⊗. Then

wt is an equivalence

$\Leftrightarrow \mathcal{C}^{\omega=0}$ is tilting

$\Leftrightarrow \mathrm{Hom}(M, N[i]) = 0 \quad \forall i \neq 0 \quad \forall M, N \in \mathcal{C}^{\omega=0}$.

Springer motives are a certain tilting family inside $DM_G(\mathcal{N})$

(Chow) motives

Define category of correspondences (over $\mathcal{N}$)

$$\mathrm{Corr}_G(\mathcal{N}) = \begin{cases} \text{objects:} & M(X/\mathcal{N}) \quad \begin{array}{l} X \text{ smooth} \\ \downarrow \text{ proper} \quad G\text{-equiv.} \\ \mathcal{N} \end{array} \\ \text{morphisms:} & \mathrm{Hom}_{\mathrm{Corr}_G(\mathcal{N})}(M(X/\mathcal{N}), M(Y/\mathcal{N})) \\ & = \mathrm{Ch}^G(X \times_{\mathcal{N}} Y) \end{cases}$$

additive category

Can take Karoubian closure $\mathrm{Kar}(\mathrm{Corr}_G(\mathcal{N}))$

e.g. $M(\mathbb{P}^1/\mathcal{N}) = \mathbb{Q} \oplus \underbrace{\mathbb{L}}_{\text{Lefschetz motive}}$

$\mathrm{Chow}_G(\mathcal{N}) := \mathrm{Kar}(\mathrm{Corr}_G(\mathcal{N}))[\mathbb{L}^{\otimes -m}]$

G-equivariant Chow motives

Bondarko: G-equiv. Chow motives form ♡ of weight structure on triangulated cat. $DM_G(\mathcal{N})$ = derived cat of G-equivariant geometric motives over $\mathcal{N}$ (= motivic sheaves on $\mathcal{N}$)

Main point behind formality theorem:

Springer motives are a certain tilting family inside $DM_G(\mathcal{N})$

Theorem (Formality)

1) Assume

(PT) $\forall x \in \mathcal{N}$ $\mathcal{M}(\mu^{-1}(x))$ is pure Tate

(FO) $\mu_i(X_i) \subseteq \mathcal{N}$ has finitely many orbits

Then $DM_G^{Spr}(\mathcal{N}, \mathbb{Q}) \xrightarrow[\text{weight complex functor}]{\text{Bondarko's}} \mathcal{D}_{perf}^{gr}(E)$

$\cap$

$DM_G(\mathcal{N}, \mathbb{Q})$ equivariant motivic sheaves on $\mathcal{N}$

2) $E_{conv} \cong E_{mot}$ as graded algebras

Applications:

- Koszul duality appears as weight complex functor
- Assumptions (PT), (FO) hold in all the previous example assuming type $\tilde{A}$ ADE in Quiver flag varieties.