September 13, 2022

Noncommutative Enriques surfaces

/$\mathbb{C}$.

I. K3

<u>Def.</u> A <u>K3 category</u> is a semiorthogonal cpt $\mathcal{C} \subset D(X)$ $(= D^b\mathrm{Coh}(X))$, X smth proj.,
w/ Serre functor $S_{\mathcal{C}} = [2]$.
$\left(\mathrm{Hom}(E,F) \cong \mathrm{Hom}(F, S_{\mathcal{C}}(E))^\vee \right)$.

<u>Examples</u> (Kuznetsov): (0) $\mathcal{C} = D(S)$, S K3 surface.

(1) $\mathcal{C} = \mathrm{Ku}(X)$, $X \subset \mathbb{P}^5$ cubic 4fold.

$D(X) = \langle \mathrm{Ku}(X), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$.

(2) $\mathcal{C} = \mathrm{Ku}(X)$, X <u>Gushel-Mukai (GM) 4fold</u>

$X = Gr(2,5) \cap \mathbb{P}^8 \cap \underbrace{Q}_{\text{quadric}} \subset \mathbb{P}^9$

or $X \xrightarrow{2:1} Gr(2,5) \cap \mathbb{P}^7$
$\cup$
$Gr(2,5) \cap \mathbb{P}^7 \cap Q$.

$D(X) = \langle \mathrm{Ku}(X), \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(1), \mathcal{U}^\vee(1) \rangle$.

(3) $\mathcal{C} = \mathrm{Ku}(X)$, X <u>Debarre-Voisin variety</u>

$X = Gr(3,10) \cap H$

$D(X) = \langle \mathrm{Ku}(X), 108$ exceptl objects$\rangle$.

⋮

Rmk: Exs are different in that $Ku(X) \not\simeq Ku(X')$ for X, X' very general of different types, but they are all deformation equivalent (still conjectural in Ex. ③)

Let X as in Ex ①+②:

Conj. (Kuz): X rational $\Longleftrightarrow$ $Ku(X) \simeq D(\text{K3 surface})$.

Conj. (Huybrechts): $Ku(X_1) \simeq Ku(X_2) \Longrightarrow X_1 \underset{bir}{\sim} X_2$.

"categorical birat'l Torelli".

Thm (BLMNPS, P-Pertusi-Zhao) $\mathcal{C} = Ku(X)$

primitive $v \in K_{num}(\mathcal{C})$ $\left(= \frac{K_0(\mathcal{C})}{\ker \chi}, \ \chi(E,F) = \sum (-1)^i \mathrm{ext}^i(E,F) \right)$

generic $\sigma \in \mathrm{Stab}^+(\mathcal{C})$

$\Rightarrow M_\sigma(v) := \left(\begin{array}{l} \text{moduli space of } \sigma\text{-stable objects} \\ \text{in } \mathcal{C} \text{ of class } v \end{array} \right)$

is nonempty iff $-\chi(v,v) + 2 \geq 0$, in which case smth proj. holomorphic symplectic of this dim.

Cor. $\mathcal{C} \simeq D(\text{K3 surface}) \Longleftrightarrow \exists\, U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \subset K_{num}(\mathcal{C})$.

II. Enriques

Def. An *Enriques category* is a semiorth cpt.
$\mathcal{D} \subset D(Y)$, Y smth proj.,
w/ $S_{\mathcal{D}} = \tau \circ [2]$ where τ generator of a $\mathbb{Z}/2$-action $\circlearrowright \mathcal{D}$.

Examples (Kuz):

⓪ $\mathcal{D} = D(T)$, T Enriques surface

$\tau = - \otimes \omega_T$. (i.e. $\omega_T^{\otimes 2} \cong \mathcal{O}_T$ and corresp. 2:1 cover $S \to T$ is K3.)

$$\boxed{D(T)^{\mathbb{Z}/2} \simeq D(S), \quad D(T) \simeq D(S)^{\mathbb{Z}/2}}$$ ← acts by involution

① $\mathcal{D} = Ku(Y)$, Y quartic double solid

$$\begin{array}{ccc} Y & \xrightarrow{2:1} & \mathbb{P}^3 \\ & & \cup \text{ quartic} \\ & & S \end{array}$$

$D(Y) = \langle Ku(Y), \mathcal{O}, \mathcal{O}(1) \rangle$ (τ = covering involution)

$$\boxed{Ku(Y)^{\mathbb{Z}/2} \simeq D(S) \quad (\text{Kuz-P.})}$$

② $\mathcal{D} = Ku(Y)$, Y GM 3fold.

$Y = Gr(2,5) \cap \mathbb{P}^7 \cap Q$.

$$\begin{array}{ccc} X & \xrightarrow{2:1} & Gr(2,5) \cap \mathbb{P}^7 \\ \text{GM 4fold.} & & \cup \\ & & Y. \end{array}$$

or

$$\begin{array}{ccc} Y & \xrightarrow{2:1} & Gr(2,5) \cap \mathbb{P}^6 \\ & & \cup \\ & & S = Gr(2,5) \cap \mathbb{P}^6 \cap Q. \end{array}$$

$D(Y) = \langle Ku(Y), \mathcal{O}, \mathcal{U}^\vee \rangle$.

$$\boxed{\begin{array}{c} Ku(Y)^{\mathbb{Z}/2} \simeq Ku(X). \\ Ku(Y)^{\mathbb{Z}/2} \simeq D(S) \end{array}}$$

③ $\mathcal{D} = Ku(Y)$, $Y \subset \mathbb{P}(1,1,1,3,3)$ degree 6.

$$D(Y) = \langle Ku(Y), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle.$$

⋮

Rmk: Exs ① – ③ are not deformation equivalent to ⓪.

Today: Focus on Exs ① + ②.

Thm 1 (Bayer-P): Y GM 3fold, Y' quartic dble solid.

(Zhang) i) $Ku(Y) \not\simeq Ku(Y')$.

ii) $Ku(Y)$ is deformation equiv. to $Ku(Y')$.

Rmk: This (dis)proves conjecture of Kuznetsov.

Thm 2 (BP): $Ku(Y_1) \simeq Ku(Y_2)$

$\Longrightarrow$ $Y_1 \underset{bir}{\sim} Y_2$ if Y_i GM 3folds

$Y_1 \cong Y_2$ if Y_i quartic dble.

Thm 3 (P-Pertusi-Zhao): $\mathcal{D} = Ku(Y)$ as in Exs ① or ②.

primitive $0 \neq v \in K_{num}(\mathcal{D}) \cong \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

$\sigma \in Stab(\mathcal{D})$ Serre invariant (i.e. σ fixed by $S_{\mathcal{D}}$ modulo $\widetilde{GL}_2^+(\mathbb{R})$)

($\exists!$ such σ up to $\widetilde{GL}_2^+(\mathbb{R})$, Pertusi....)

$\Rightarrow$ $M_\sigma(v)$ nonempty, smth of dim $-\chi(v,v)+1$
at $E \in M_\sigma(v)$ not fixed by τ.

III. K3 covers + proofs :

Thm (Elagin): $\mathcal{D} \circlearrowleft \mathbb{Z}/2$ via τ Enriques cat.

$$\mathcal{C} := \mathcal{D}^{\mathbb{Z}/2} = \left\{ (E,\phi) \;\middle|\; E \in \mathcal{D},\ \phi : E \xrightarrow{\sim} \tau(E) \text{ s.t. } \phi^2 = \mathrm{id} \right\}$$

1) $\mathcal{C}$ is K3 category. (K3 cover of $\mathcal{D}$).

2) $\exists$ residual action $\mathbb{Z}/2 \circlearrowleft \mathcal{C}$ s.t.
$$\mathcal{D} \simeq \mathcal{C}^{\mathbb{Z}/2}.$$

$(\mathbb{Z}/2 = \{\pm 1\} \ni \chi, \quad \chi \cdot (E,\phi) = (E, \chi\cdot\phi))$

Examples: In blue boxes above

Pf Thm 1 ii):

$Y \xrightarrow{2:1} Gr(2,5) \cap \mathbb{P}^6$ GM , $Y' \longrightarrow \mathbb{P}^3$ quartic dble.
$\qquad\qquad\quad \cup \qquad\qquad\qquad\qquad\qquad\qquad \cup$
$\qquad\qquad\quad S. \qquad\qquad\qquad\qquad\qquad\qquad S'$

$Ku(Y) \simeq D(S)^{\mathbb{Z}/2}$. $\qquad Ku(Y') \simeq D(S')^{\mathbb{Z}/2}$.

Find specialization of K3s
$$S \rightsquigarrow S'.$$

compatible w/ $\mathbb{Z}/2$ actions on $D(S)$, $D(S')$.

Take $\mathbb{Z}/2$-invariants. $\blacksquare$