Bounded t-Structures on the category of perfect complexes

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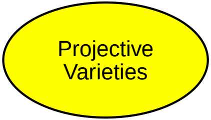
Overview

1. The papers that introduced me to the subject

2. t-structures: example and formal definition

3. Something about the proof

Mumford's view of algebraic geometry, in pictures



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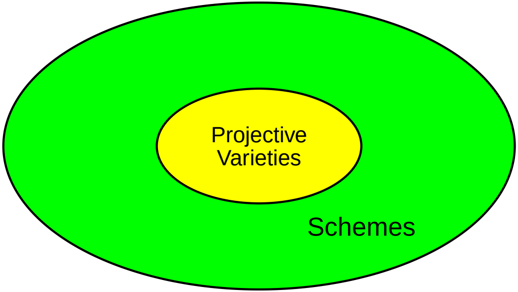

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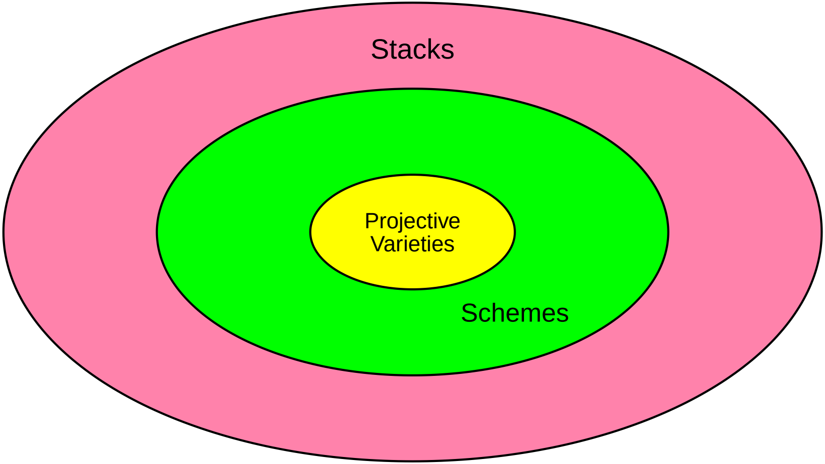

My introduction to noncommutative geometry

Dmitri O. Orlov, *Smooth and proper noncommutative schemes and gluing of DG categories*, Adv. Math. **302** (2016), 59–105.

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Example (the standard t–structure on $\mathbf{D}(\mathcal{A})$)

Let $\mathcal{A}$ be an abelian category. We define two full subcategories of $\mathbf{D}(\mathcal{A})$:

- $$\mathbf{D}(\mathcal{A})^{\leq 0} \quad = \quad \{A^* \in \mathbf{D}(\mathcal{A}) \mid H^i(A^*) = 0 \text{ for all } i > 0\}$$
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Put $I = \mathrm{Im}(Y^{-1} \to Y^0)$, and $Q = Y^0/I$.

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A t–structure on a triangulated category $\mathcal{T}$ is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying

- $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq 0}[1]$
- $\mathrm{Hom}\Big(\mathcal{T}^{\leq 0}[1]\,,\ \mathcal{T}^{\geq 0}\Big) = 0$
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Given an object $B \in \mathcal{T}$, the third property of a t-structure says that there exists an exact triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

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Given an object $B \in \mathcal{T}$, the third property of a t-structure says that there exists an exact triangle

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with $A \in \mathcal{T}^{\leq 0}[1]$ and with $C \in \mathcal{T}^{\geq 0}$.

This triangle is often written

$$B^{\leq -1} \longrightarrow B \longrightarrow B^{\geq 0} \longrightarrow B^{\leq -1}[1]$$

Notation

For $n \in \mathbb{Z}$ we adopt the shorthand

$$\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n] , \qquad \mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n] .$$

Definition (Bounded t-Structures)

A t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called bounded if, for every object $X \in \mathcal{T}$, there exists an integer $n > 0$ with

$$X[n] \in \mathcal{T}^{\leq 0} \qquad \text{and} \qquad X[-n] \in \mathcal{T}^{\geq 0} .$$

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Let X be a scheme.

Example

1. $\mathbf{D}_{\mathbf{qc}}(X)$ will be our shorthand for $\mathbf{D}_{\mathbf{qc}}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of sheaves of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.
2. The objects of $\mathbf{D}^{\mathrm{perf}}(X)$ are the perfect complexes. A complex is *perfect* if it is locally isomorphic to a bounded complex of vector bundles.
3. Assume X is noetherian. The objects of $\mathbf{D}^{b}_{\mathbf{coh}}(X)$ are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

Let X be a scheme, and $Z \subset X$ a closed subset.

Example

1. $\mathbf{D}_{\mathbf{qc},Z}(X) \subset \mathbf{D}_{\mathbf{qc}}(X)$ will be the full subcategory with objects the complexes whose restriction to $X - Z$ is acyclic.
2. $\mathbf{D}^{\mathrm{perf}}_Z(X) \subset \mathbf{D}^{\mathrm{perf}}(X)$ will be the full subcategory with objects the complexes whose restriction to $X - Z$ is acyclic.
3. Assuming X is noetherian, $\mathbf{D}^b_{\mathbf{coh},Z}(X) \subset \mathbf{D}^b_{\mathbf{coh}}(X)$ will be the full subcategory with objects the complexes whose restriction to $X - Z$ is acyclic.

Self-dual

The definition of a t-structure is self-dual. If $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure on $\mathcal{T}$ then $\left((\mathcal{T}^{\geq 0})^{\mathrm{op}}, (\mathcal{T}^{\leq 0})^{\mathrm{op}}\right)$ is a t-structure on $\mathcal{T}^{\mathrm{op}}$.

Self-dual

The definition of a *t*-structure is self-dual. If $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ is a *t*-structure on $\mathcal{T}$ then $\Big(\big(\mathcal{T}^{\geq 0}\big)^{\mathrm{op}}, \big(\mathcal{T}^{\leq 0}\big)^{\mathrm{op}}\Big)$ is a *t*-structure on $\mathcal{T}^{\mathrm{op}}$.

The t-structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ is bounded on $\mathcal{T}$ if and only if the dual t-structure is bounded on $\mathcal{T}^{\mathrm{op}}$.

Conjecture

Let X be a finite-dimensional, noetherian scheme. The category $\mathbf{D}^{\mathrm{perf}}(X)$ has a bounded t-structure if and only if X is regular, in which case $\mathbf{D}^{\mathrm{perf}}(X) = \mathbf{D}^{b}_{\mathbf{coh}}(X)$.

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This can be found as Conjecture 1.5 in

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Let $\mathcal{M}$ be a model category with homotopy category $\mathcal{T}$, and assume $\mathcal{T}$ has a bounded t–structure. Antieau, Gepner and Heller proved:

1. If the abelian category $\mathcal{T}^{\heartsuit}$ is noetherian, then $K_n(\mathcal{M}) = 0$ for $n < 0$.
2. Unconditionally we have $K_{-1}(\mathcal{M}) = 0$.

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If $\mathcal{A}$ is an abelian category, and $\mathcal{T} = \mathbf{D}^b(\mathcal{A})$ with the usual model structure, the vanishing in negative K-theory is due to Schlichting.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Corollary

Let X be a finite-dimensional, noetherian, separated scheme. Assume $K_{-1}(X)$ is nonzero. Then the category $\mathbf{D}^{\mathrm{perf}}(X)$ has no bounded t-structure.

If $K_n(X)$ is nonzero for $n \leq -2$, then any bounded t-structure on $\mathbf{D}^{\mathrm{perf}}(X)$ cannot have a noetherian heart.

Corollary

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This can be found as Corollary 1.4 in

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Theorem

Let X be a a scheme, and let $Z \subset X$ be a closed subset. Let $\mathbf{D}_Z^{\mathrm{perf}}(X)$ be the derived category whose objects are the perfect complexes on X whose restriction to $X - Z$ is acyclic.

Now assume X is noetherian and finite-dimensional. Then the category $\mathbf{D}_Z^{\mathrm{perf}}(X)$ has a bounded t-structure if and only if Z is contained in the regular locus of X, in which case $\mathbf{D}_Z^{\mathrm{perf}}(X) = \mathbf{D}^b_{\mathbf{coh},Z}(X)$.

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For the proof see

Amnon Neeman, *Bounded t–structures on the category of perfect complexes*, https://arxiv.org/abs/2202.08861.

Something about the proof

It suffices to show that the inclusion $\mathbf{D}_Z^{\mathrm{perf}}(X) \longrightarrow \mathbf{D}^b_{\mathbf{coh},Z}(X)$ is an equivalence.

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Take $F \in \mathbf{D}^b_{\mathbf{coh},Z}(X)$. Without loss of generality assume $F \in \mathbf{D}^b_{\mathbf{coh},Z}(X)^{\geq 0}$. We want to show that $F \in \mathbf{D}_Z^{\mathrm{perf}}(X)$.

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Resolving F by vector bundles, we may represent it as a complex

$$\cdots \longrightarrow \mathcal{V}^{m-1} \longrightarrow \mathcal{V}^{m} \longrightarrow \cdots \longrightarrow \mathcal{V}^{n-1} \longrightarrow \mathcal{V}^{n} \longrightarrow 0 \longrightarrow \cdots$$

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\end{array}$$

This gives an exact triangle

$$D \longrightarrow E \longrightarrow F \longrightarrow D[1],$$

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- Joseph Lipman and Amnon Neeman, *Quasi-perfect scheme maps and boundedness of the twisted inverse image functor*, Illinois J. Math. **51** (2007), 209–236.

For a proof that works in the relative context, that is given $F \in \mathbf{D}^b_{\mathbf{coh},Z}(X)$ it produces a triangle

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with $E \in \mathbf{D}^{\mathrm{perf}}_Z(X)$ and $D \in \mathbf{D}^b_{\mathbf{coh},Z}(X)^{\leq m}$, see

Tag 36.14 in the Stacks Project.

Anyway: given any object $F \in \mathbf{D}^b_{\mathbf{coh},Z}(X)^{\geq 0}$ and any integer $m \ll 0$, we produced an exact triangle

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Now: in the category $\mathbf{D}^b_{\mathbf{coh},Z}(X)$ there is a standard t-structure, and we may form truncations with respect to shifts of it. This gives, for every integer $\ell \in \mathbb{Z}$, a triangle

$$E^{\leq \ell} \longrightarrow E \longrightarrow E^{\geq \ell+1} \longrightarrow E^{\leq \ell}[1].$$

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The choice of E guarantees that, for $m < \ell < 0$, this triangle coincides with $D \longrightarrow E \longrightarrow F \longrightarrow D[1]$.

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$$D \longrightarrow E \longrightarrow F \longrightarrow D[1],$$

with $E \in \mathbf{D}^{\mathrm{perf}}_Z(X)$ and $D \in \mathbf{D}^b_{\mathbf{coh},Z}(X)^{\leq m}$.

Now: in the category $\mathbf{D}^b_{\mathbf{coh},Z}(X)$ there is a standard t-structure, and we may form truncations with respect to shifts of it. This gives, for every integer $\ell \in \mathbb{Z}$, a triangle

$$E^{\leq \ell} \longrightarrow E \longrightarrow E^{\geq \ell+1} \longrightarrow E^{\leq \ell}[1].$$

The choice of E guarantees that, for $m < \ell < 0$, this triangle coincides with $D \longrightarrow E \longrightarrow F \longrightarrow D[1]$.

Now assume that the category $\mathbf{D}^{\mathrm{perf}}_Z(X)$ has a bounded t-structure.

Definition

Let $\mathcal{T}$ be a triangulated category. Two t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ are declared equivalent if there exists an integer $n > 0$ with

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We are given a bounded t-structure on $\mathbf{D}_Z^{\mathrm{perf}}(X)$, and we would like to compare it to the standard t-structure on $\mathbf{D}^b_{\mathbf{coh},Z}(X)$. For technical reasons this is easiest to do in $\mathbf{D}_{\mathbf{qc},Z}(X)$.

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We appeal to Theorem A.1 in

Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio, *Construction of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2523–2543 (electronic).

Theorem

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{A} \subset \mathcal{T}$ be a set of compact objects satisfying $\mathcal{A}[1] \subset \mathcal{A}$.

Let $\mathrm{Coprod}(\mathcal{A})$ be the smallest full subcategory of $\mathcal{T}$, containing $\mathcal{A}$ and closed under coproducts and extensions.

Then $\left(\mathrm{Coprod}(\mathcal{A}), \mathrm{Coprod}(\mathcal{A})[1]^{\perp}\right)$ is a t-structure on $\mathcal{T}$.

This is Theorem A.1 in

Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio, *Construction of t-structures and equivalences of derived categories*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2523–2543 (electronic).

It suffices to show that the standard t-structure on $\mathbf{D}_{\mathbf{qc},Z}(X)$ is equivalent to the t-structure generated by $\mathcal{A} = \mathbf{D}_Z^{\mathrm{perf}}(X)^{\leq 0}$, where generation is in the sense of Alonso, Jeremías and Souto.

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We need to prove the inclusion

$$\mathbf{D}_{\mathbf{qc},Z}(X)^{\leq 0} \quad \subset \quad \mathrm{Coprod}(\mathcal{A}[-n])$$

for some integer n.

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for some integer n.

Following Mumford, we pay particular attention to the case where X is a projective variety.

Pick any object $F \in \mathbf{D_{qc}}(X)^{\leq 0}$. Resolving it, we may produce an isomorph

$$\cdots \longrightarrow \mathcal{V}^{m-1} \longrightarrow \mathcal{V}^{m} \longrightarrow \cdots \longrightarrow \mathcal{V}^{-1} \longrightarrow \mathcal{V}^{0} \longrightarrow 0 \longrightarrow \cdots$$

where each $\mathcal{V}^i$ is a coproduct of line bundles $\mathcal{O}(-\ell)$ for $\ell > 0$.

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Now $\mathcal{A} = \mathbf{D}^{\mathrm{perf}}(X)^{\leq 0}$ for a bounded t-structure

$$\left(\mathbf{D}^{\mathrm{perf}}(X)^{\leq 0}, \mathbf{D}^{\mathrm{perf}}(X)^{\geq 0}\right)$$

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on the category $\mathbf{D}^{\mathrm{perf}}(X)$. Hence, given any integer $N > 0$, we can find an integer $M > 0$ such that

$$\mathcal{O}(-\ell) \in \mathcal{A}[-M] \qquad \text{for all } 0 \leq \ell \leq N.$$

Alexei I. Bondal and Michel Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

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Dmitri O. Orlov, *Smooth and proper noncommutative schemes and gluing of DG categories*, Adv. Math. **302** (2016), 59–105.

Let R be a commutative ring. The short exact sequence

$$0 \longrightarrow R[x] \xrightarrow{\ x\ } R[x] \longrightarrow R \longrightarrow 0$$

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Tensoring together $n+1$ of these we deduce a quasi-isomorphism of R with the Koszul complex

$$\bigotimes_{i=0}^{n} \left(R[x_i] \xrightarrow{X_i} R[x_i] \right)$$

Applying Proj to this, we obtain a quasi-isomorphism of $\mathcal{O}(1)$ with a complex

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \oplus\mathcal{O}(-n+1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(-1) \longrightarrow \oplus\mathcal{O} \longrightarrow 0$$

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Tensoring this with itself $\ell > 0$ times yields a quasi-isomorphism of $\mathcal{O}(\ell)$ with some complex

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And this brutal truncation must be quasi-isomorphic to $\mathcal{O}(\ell) \oplus \mathcal{V}[n]$ for some vector bundle $\mathcal{V}$.

Applying the functor $(-)^\vee = \mathcal{RH}om(-, \mathcal{O})$, we obtain a quasi-isomorphism of $\mathcal{O}(-\ell) \oplus \mathcal{V}^\vee[-n]$ with

$$0 \longrightarrow \oplus\mathcal{O} \longrightarrow \oplus\mathcal{O}(1) \longrightarrow \cdots \longrightarrow \oplus\mathcal{O}(n-1) \longrightarrow \oplus\mathcal{O}(n) \longrightarrow 0$$

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Thus if $\mathcal{A}[-M]$ contains

$$\mathcal{O}, \quad \mathcal{O}(1)[-1], \quad \ldots, \quad \mathcal{O}(n-1)[-n+1], \quad \mathcal{O}(n)[-n]$$

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But then

$$\mathbf{D_{qc}}(X)^{\leq 0} \quad \subset \quad \text{Coprod}(\mathcal{A}[-M])\ .$$

Amnon Neeman, *Strong generators in* $\mathbf{D}^{\mathrm{perf}}X$ *and* $\mathbf{D}^b_{\mathbf{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

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Thank you!