

Non commutative Shapes, a conference in honour of Michel Van den Bergh,

Antwerp, Wednesday, September 14, 9-9.50 am

Notes at bit.ly/kellernotes

On Amiot's conjecture

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Aim: Present an application of VdB's superpotential theorem (2015) to Amiot's conjecture

Plan: 1. From preprojective algebras to Amiot's conjecture

2. VdB's superpotential theorem

3. The application

1. From preprojective algebras to Amiot's conjecture

Δ an ADE Dynkin diagram, \mathcal{Q} an orientation of Δ , e.g. $\mathcal{Q} = \vec{A}_3: 1 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3$

$k = \mathbb{C}$ for simplicity

$\Pi(Q)$ the preprojective algebra (Gelfand-Ponomarev 1976) of Q over k , e.g.

$$1 \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\beta^*} \end{array} 2 \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\alpha^*} \end{array} 3 \quad \text{with } g = \sum_{\gamma \in Q_1} [\gamma, \gamma^*] \text{ or with } -\beta^*\beta = 0, \beta\beta^* - \alpha\alpha^* = 0, \alpha\alpha^* = 0.$$

↓ right

Rks: $\Pi(Q)$ is finite-dimensional and selfinjective (injective as a module over itself)

so $\text{mod } \Pi(Q) = \{ \text{fin. dim. right } \Pi(Q)\text{-modules} \}$ is a Frobenius category.

mod $\Pi(Q) = (\text{mod } \Pi(Q)) / (\text{proj. - inj.})$, Hom, is triangulated (Happel 1986).

Rks: 1) mod $\Pi(Q)$ is 2-Calabi-Yau as a triang. category (Crawley-Boevey, 2000), i.e. we have

$$\overset{k\text{-dual}}{\rightarrow} D\text{Ext}^1(L, M) \cong \text{Ext}^1(M, L), \quad \forall L, M \in \text{mod } \Pi(Q).$$

2) $\Pi(Q)$ is wild except if $\Delta \in \{A_1, A_2, A_3, A_4, D_4, A_5\}$.

3) $\Pi(Q)$ is always 2-representation-finite (in the sense of Iyama 2007),

i.e. $\underline{\text{mod}} \Pi(B)$ contains a (canonical) cluster-tilting object T_{can} .

equivalently: $\underline{\text{mod}} \Pi(B)$

constructed by Geiss-Leclerc-Schröer (2006, 2009)

Def. (Iyama 2009): $T \in \underline{\text{mod}} \Pi(B)$ is (2-) cluster-tilting if

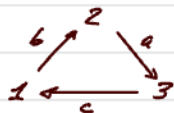
a) T is rigid, i.e. $\text{Ext}^2(T, T) = 0$

b) T is a 2-step generator of $\underline{\text{mod}} \Pi(B)$, i.e. $\forall M \in \underline{\text{mod}} \Pi(B)$, there is a triangle

$$T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow \Sigma T_1$$

with $T_0, T_1 \in \text{add}(T)$.

Example: $\mathbb{Q} : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. Then $\text{End}_{\mathbb{Q}}(T)$ is given by



with relations $ab=0$, $bc=0$, $ca=0$.

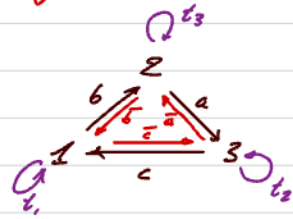
This means that $\text{End}_e(T) = \mathcal{J}_{R,W} = \text{Jacobian algebra of } (R,W)$, where

$$R: \begin{array}{ccc} & 2 & \\ b \nearrow & & \searrow a \\ 1 & \xleftarrow{c} & 3 \end{array}, \quad W = abc \quad (\leadsto \text{relations } \partial_c W = ab, \partial_a W = bc, \partial_b W = ca)$$

The Jacobian algebra has a dg refinement: the Ginzburg dg algebra $\Gamma_{R,W}$

$\Gamma_{R,W} = \text{completed graded path alg. of } \tilde{R}$:

with d s.th. $dt_i = c\bar{c} - \bar{b}b, \dots, d\bar{a} = \partial_a W = bc, \dots$



$$|t_i| = -2$$

$$|\bar{a}| = |\bar{b}| = |\bar{c}| = -1$$

Thm (Amiot 2009): We have a canonical triangle equivalence

$$\mathcal{C}_{R,W} \xrightarrow{\sim} \underline{\text{mod}} \text{TT}(\mathcal{B}), \quad \Gamma_{R,W} \xrightarrow{1} T_{\text{can}}$$

where $\mathcal{C}_{R,W}$ is the (generalized) cluster category

$$\mathcal{C}_{R,W} = \text{per}(\Gamma_{R,W}) / \text{pvd}(\Gamma_{R,W}).$$

perfect derived category
thick $(\Gamma_{R,W}) \subseteq \mathcal{DT}_{R,W}$

perfectly valued der. cat.
 $\{M \in \mathcal{DT} \mid M_k \in \text{perf } k\}$

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Conjecture (Amiot 2010): Let \mathcal{C} be a Hom-finite, Kaloubian, triang. cat. s.th.

- \mathcal{C} is algebraic (i.e. $\mathcal{C} \stackrel{\cong}{=} H^0 \mathcal{A}$ for some pretriang. dg cat. \mathcal{A})
- \mathcal{C} is 2-Calabi-Yau as a triang. cat.
- \mathcal{C} contains a cluster-tilting object T .

Then $\exists (R, W)$ s.th. $\mathcal{C}_{R,W} \xrightarrow{\cong} \mathcal{C}$, $\Gamma_{R,W} \hookrightarrow T$.

In particular, we have $\Gamma_{R,W} = \text{End}_{\mathcal{C}}(T)$.

Evidence: 1) Ok if $\text{End}(T)$ is hereditary (K-Reiten '08).

2) Ok if $\mathcal{C} = \text{mod } \Pi(10)$, $T = T_{\text{can}}$, cf. above

3) Ok if $\mathcal{C} = \underline{\text{CM}}^G(R)$, $R = k[x,y,z]$, G suitable cyclic (Amiot-Iyama-Reiten 2011, Thanhoffer-VdB '15)

Incoherence: The CY-structure should be given on \mathcal{A} , not on $H^0 \mathcal{A} \cong \mathbb{C}$!

Thm (K-Liu): After this modification, the conjecture holds.

2. VdB's superpotential theorem (in dimension 3)

Thm (VdB '15): Let A be a smooth connective complete dg algebra endowed with a left 3-CY structure.

Then A is quasi-isom. to $T_{R,W}$ for a quiver with potential (R,W) .

Rks: The converse also holds (VdB '11).

Terminology: smooth: $A \in \text{per}(A^e)$, $A^e = A \otimes A^{\text{op}}$, connective: $H^p A = 0$, $\forall p > 0$

complete: pseudo-compact + ... [holds for completed connective dg path algebras]

left 3-CY structure: class $\beta \in \text{HH}_3(A)$ whose image under

$$\text{HH}_3(A) \rightarrow \text{HH}_3(A) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}A^e}(A^\vee, \Sigma^{-3}A)$$

is an isomorphism, where $A^\vee = \text{RHom}_{A^e}(A, A^e)$.

3. Application to Amiot's modified conjecture

\mathcal{C} Hom-finite algebraic, with a 2-cluster-tilting object T and a right 2-CY structure.

i.e. $\alpha \in \text{DHC}_{-2}(\mathcal{C}_{\mathcal{C}_2})$ which is non deg., i.e. its image in

$$\text{DHH}_{-2}(\mathcal{C}_{\mathcal{C}_2}) \simeq \text{Hom}_{\mathcal{D}(\mathcal{C}_{\mathcal{C}_2}^e)}(\mathcal{C}_{\mathcal{C}_2}, \Sigma^{-2}D\mathcal{C}_{\mathcal{C}_2}^{\text{op}})$$

is an isomorphism.

To construct: (R, W) s.t. we have an exact sequence

$$0 \rightarrow \boxed{\text{pvd}(\Gamma_{R,W}) \rightarrow \text{per}(\Gamma_{R,W}) \rightarrow \mathcal{C} \rightarrow 0.}$$

$\downarrow \cong$
 $\text{per}(\Gamma_{R,W}')$

Know: $\mathcal{C} = \underline{\mathcal{E}} = \mathcal{E}/(\mathcal{P})$, \mathcal{E} Frobenius cat.

$\mathcal{P} \subseteq \mathcal{E}$ subcat. of proj.-inj.

$$\begin{array}{ccc} \mathcal{C} & \longleftarrow & \mathcal{E} \\ \cup & & \cup \\ \text{addT} & \longleftarrow & \mathcal{M} \end{array}$$

We have the diagram:

$$\begin{array}{ccccc} & & \mathcal{H}^b(\mathcal{P}) = \mathcal{H}^b(\mathcal{P}) & & \\ & & \downarrow & & \downarrow \\ \mathcal{H}_{ac}^b(\mathcal{M}) \hookrightarrow & \mathcal{H}^b(\mathcal{M}) & \longrightarrow & \mathcal{D}^b(\mathcal{E}) & \\ \downarrow & \downarrow & & \downarrow & \\ \mathcal{H}_{ac}^b(\mathcal{M}) \hookrightarrow & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) & \longrightarrow & \underline{\mathcal{E}} = \mathcal{C} & \\ \downarrow & \downarrow & & & \\ \text{per}(\mathcal{P}!) & \text{per} \mathcal{P} & & & \end{array}$$

Get a triangle:

$$\underbrace{HC(\Gamma)}_{\leq 0} \rightarrow HC(\mathcal{C}_2) \rightarrow \Sigma HC(\Gamma') \rightarrow \underbrace{\Sigma HC(\Gamma)}_{\leq -1}$$

$$\begin{array}{ccc} DHC_{-2}(\mathcal{C}_2) & \xrightarrow{\cong} & DHC_{-3}(\Gamma') \\ \downarrow \alpha & & \downarrow \beta \\ & & HN_3(\Gamma) \\ & \searrow & \downarrow \psi \\ & & \beta \end{array}$$

Subtle point: α non deg. \Rightarrow β non deg. !

VdB's superpot. thm \Rightarrow $\Gamma \cong \Gamma_{R,W}$ for some (R,W) .