Noncommutative Shapes, a conference in honour of Michel Van den Bergh,

Antwerp, Wednesday, September 14, 9–9.50 am

Notes at bit.ly/kellersnotes

On Amiot's conjecture

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Aim: Present an application of VdB's superpotential theorem (2015) to Amiot's conjecture

Plan:
1. From preprojective algebras to Amiot's conjecture
2. VdB's superpotential theorem
3. The application

1. From preprojective algebras to Amiot's conjecture

Δ an ADE Dynkin diagram, Q an orientation of Δ, e.g. $Q = \vec{A}_3 : 1 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3$

$k = \mathbb{C}$ for simplicity

$\Pi(Q)$ the preprojective algebra (Gelfand-Ponomarev 1976) of Q over k, e.g.

$1 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 2 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 3$ with $\rho = \sum_{\gamma \in Q_1} [\gamma, \gamma^*]$ or with $-\beta^*\beta = 0, \beta\beta^* - \alpha\alpha^* = 0, \alpha\alpha^* = 0.$

Rk: $\Pi(Q)$ is finite-dimensional and selfinjective (injective as a right module over itself)

so $\operatorname{mod}\Pi(Q) = \{$ fin. dim. right $\Pi(Q)$-modules $\}$ is a Frobenius category.

$\underline{\operatorname{mod}}\,\Pi(Q) = (\operatorname{mod}\Pi(Q)) / (\text{proj.-inj.}), \underline{\operatorname{Hom}}$, is triangulated (Happel 1986).

Rks: 1) $\underline{\operatorname{mod}}\,\Pi(Q)$ is 2-Calabi-Yau as a triang. category (Crawley-Boevey, 2000), i.e. we have

k-dual

$$D\operatorname{Ext}^1(L,M) \simeq \operatorname{Ext}^1(M,L), \quad \forall L, M \in \underline{\operatorname{mod}}\,\Pi(Q).$$

2) $\Pi(Q)$ is wild except if $\Delta \in \{A_1, A_2, A_3, A_4, D_4, A_5\}$.

3) $\Pi(Q)$ is always 2-representation-finite (in the sense of Iyama 2007),

i.e. $\underline{\mathrm{mod}}\,\Pi(Q)$ contains a (canonical) cluster-tilting object T_{can}.

equivalently: $\mathrm{mod}\,\Pi(Q)$ constructed by Geiss-Leclerc-Schröer (2006, 2007)

Def. (Iyama 2007): $T \in \underline{\mathrm{mod}}\,\Pi(Q)$ is (2-) cluster-tilting if

a) T is rigid, i.e. $\mathrm{Ext}^1(T,T) = 0$

b) T is a 2-step generator of $\underline{\mathrm{mod}}\,\Pi(Q)$, i.e. $\forall M \in \underline{\mathrm{mod}}\,\Pi(Q)$, there is a triangle

$$T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow \Sigma T_1$$

with $T_0, T_1 \in \mathrm{add}(T)$.

Example: $Q: 1 \to 2 \to 3 \to 4$. Then $\underline{\mathrm{End}}(T)$ is given by

$$\begin{tikzcd} & 2 \arrow[dr, "a"] & \\ 1 \arrow[ur, "b"] & & 3 \arrow[ll, "c"] \end{tikzcd}$$

with relations $ab = 0$, $bc = 0$, $ca = 0$.

This means that $\mathrm{End}_{\mathcal{C}}(T) = J_{R,W}$ = Jacobian algebra of (R,W), where

$$R: \begin{tikzcd} & 2 \arrow[dr, "a"] & \\ 1 \arrow[ur, "b"] & & 3 \arrow[ll, "c"] \end{tikzcd} \quad , \quad W = abc \quad (\rightsquigarrow \text{ relations } \partial_c W = ab,\ \partial_a W = bc,\ \partial_b W = ca)$$

The Jacobian algebra has a dg refinement: the Ginzburg dg algebra $\Gamma_{R,W}$

$\Gamma_{R,W}$ = completed graded path alg. of $\tilde{R}$:

$$\begin{tikzcd} & 2 \arrow[loop above, "t_3"] \arrow[dr, shift left, "a"] \arrow[dl, shift left, "\bar{b}"] & \\ 1 \arrow[loop left, "t_1"] \arrow[ur, shift left, "b"] \arrow[rr, shift right, "\bar{c}"'] & & 3 \arrow[loop right, "t_2"] \arrow[ul, shift left, "\bar{a}"] \arrow[ll, shift right, "c"'] \end{tikzcd}$$

$|t_i| = -2$

$|\bar{a}| = |\bar{b}| = |\bar{c}| = -1$

with d s.th. $dt_1 = c\bar{c} - \bar{b}b, \ldots,\ d\bar{a} = \partial_a W = bc, \ldots$

Thm (Amiot 2009): We have a canonical triangle equivalence

$$\mathcal{C}_{R,W} \xrightarrow{\sim} \underline{\mathrm{mod}}\, \Pi(Q), \quad \Gamma_{R,W} \longmapsto T_{can}$$

where $\mathcal{C}_{R,W}$ is the (generalized) cluster category

$$\mathcal{C}_{R,W} = \mathrm{per}(\Gamma_{R,W}) / \mathrm{pvd}(\Gamma_{R,W}).$$

perfect derived category
$\mathrm{thick}(\Gamma_{R,W}) \subseteq \mathcal{D}\Gamma_{R,W}$

perfectly valued der. cat.
$\{M \in \mathcal{D}\Gamma \mid M|_k \in \mathrm{per}\,k\}$

Conjecture (Amiot 2010): Let $\mathcal{C}$ be a Hom-finite, Karoubian, triang. cat. s.th.

a) $\mathcal{C}$ is algebraic (i.e. $\mathcal{C} \xrightarrow[\Delta]{\sim} H^0\mathcal{A}$ for some pretriang. dg cat. $\mathcal{A}$)

b) $\mathcal{C}$ is 2-Calabi-Yau as a triang. cat.

c) $\mathcal{C}$ contains a cluster-tilting object T.

Then $\exists\ (R, W)$ s.th. $\mathcal{C}_{R,W} \xrightarrow{\sim} \mathcal{C}$, $\Gamma_{R,W} \mapsto T$.

In particular, we have $\mathcal{J}_{R,W} \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}}(T)$.

Evidence: 1) Ok if $\mathrm{End}(T)$ is hereditary (K-Reiten '08).

2) Ok if $\mathcal{C} = \underline{\mathrm{mod}}\,\Pi(Q)$, $T = T_{can}$, cf. above

3) Ok if $\mathcal{C} = \underline{\mathrm{CM}}^G(R)$, $R = k[[x, y, z]]$, G suitable cyclic (Amiot-Iyama-Reiten 2011, Thanhoffer-VdB '15)

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Incoherence: The CY-structure should be given on $\mathcal{A}$, not on $H^0\mathcal{A} \cong \mathbb{C}$!

Thm (K-Liu): After this modification, the conjecture holds.

2. VdB's superpotential theorem (in dimension 3)

Thm (VdB '15): Let A be a smooth connective complete dg algebra endowed with a left 3-CY structure. Then A is quasi-isom. to $\Gamma_{R,W}$ for a quiver with potential (R,W).

Rks: The converse also holds (VdB '11).

Terminology: smooth: $A \in \mathrm{per}(A^e)$, $A^e = A \otimes A^{op}$, connective: $H^pA = 0$, $\forall p > 0$

complete: pseudo-compact + ... [holds for completed connective dg path algebras] left 3-CY structure: class $\beta \in HN_3(A)$ whose image under

$$HN_3(A) \longrightarrow HH_3(A) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A^e}(A^\vee, \Sigma^{-3}A)$$

is an isomorphism, where $A^\vee = R\mathrm{Hom}_{A^e}(A, A^e)$.

3. Application to Amiot's modified conjecture

$\mathcal{C}$ Hom-finite algebraic, with a 2-cluster-tilting object T and a right 2-CY structure,

i.e. $\alpha \in DHC_{-2}(\mathcal{C}_{dg})$ which is non deg., i.e. its image in

$$DHH_{-2}(\mathcal{C}_{dg}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{C}_{dg}^e)}(\mathcal{C}_{dg}, \Sigma^{-2} D\, \mathcal{C}_{dg}^{op})$$

is an isomorphism.

To construct: (R, W) s.th. we have an exact sequence

$$0 \rightarrow \boxed{\mathrm{pvd}(\Gamma_{R,W}) \rightarrow \mathrm{per}(\Gamma_{R,V}) \longrightarrow \mathcal{C}} \longrightarrow 0.$$

$$\begin{array}{c} | \wr \\ \mathrm{per}(\Gamma^!_{R,V}) \end{array}$$

Know: $\mathcal{C} = \underline{\mathcal{E}} = \mathcal{E}/(\mathcal{P})$, $\mathcal{E}$ Frobenius cat.

$\mathcal{P} \subseteq \mathcal{E}$ subcat. of proj.-inj.

$$\begin{tikzcd}
\mathcal{C} & \mathcal{E} \arrow[l, two heads] \\
\cup & \cup \\
\operatorname{add} T & \mathcal{M} \arrow[l, two heads] \arrow[ul, phantom, "\ulcorner" near start]
\end{tikzcd}$$

We have the diagram:

$$\begin{tikzcd}
 & \mathcal{H}^b(\mathcal{P}) \arrow[r, equal] \arrow[d] & \mathcal{H}^b(\mathcal{P}) \arrow[d] \\
\mathcal{H}^b_{ac}(\mathcal{M}) \arrow[r, hook] \arrow[d, equal] & \mathcal{H}^b(\mathcal{M}) \arrow[r] \arrow[d] & \mathcal{D}^b(\mathcal{E}) \arrow[d] \\
\mathcal{H}^b_{ac}(\mathcal{M}) \arrow[r, hook] \arrow[d, no head, "\wr"] & \mathcal{H}^b(\mathcal{M})/\mathcal{H}^b(\mathcal{P}) \arrow[r] \arrow[d, no head, "\wr"] & \underline{\mathcal{E}} = \mathcal{C} \\
\operatorname{per}(\Gamma^!) & \operatorname{per}\Gamma &
\end{tikzcd}$$

Get a triangle:

$$\underbrace{HC(\Gamma)}_{\leq 0} \to HC(\mathcal{C}_{dg}) \to \Sigma HC(\Gamma^!) \to \underbrace{\Sigma HC(\Gamma)}_{\leq -1}$$

$$\begin{tikzcd} DHC_{-2}(\mathcal{C}_{dg}) & DHC_{-3}(\Gamma^!) \arrow[l, "\sim"'] \\ \ni \alpha \arrow[dr, bend right, mapsto] & HN_3(\Gamma) \arrow[u, "\wr"'] \\ & \ni \beta \end{tikzcd}$$

Subtle point: α non deg. $\Longrightarrow$ β non deg. !

VdB's superpol. thm $\Longrightarrow$ $\Gamma \cong \Gamma_{R,W}$ for some (R, W).