The noncommutative minimal model program

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Outline

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Structure of derived categories

What is fascinating about the bounded derived category of coherent sheaves $D^b(X)$ on a smooth projective variety X?

Hidden structure

Example (D-equivalence conjecture)

$D^b(X) \cong D^b(X')$ for birationally equivalent projective Calabi-Yau manifolds.

Example (Beilinson's theorem)

$D^b(\mathbb{P}^n)$ admits a full exceptional collection $\mathscr{O}, \mathscr{O}(1), \ldots, \mathscr{O}(n)$.

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Stucture of derived categories

An elaboration of Beilinson's theorem:

Example (Dubrovin's conjecture)

A smooth Fano variety has a full exceptional collection if and only if its big quantum cohomology is generically semisimple.

Also some failed hopes: the existence of phantom categories.

Example (Barlow surfaces)

Have exceptional collections of line bundles $L_1, \ldots, L_{11} \in D^b(X)$ that span $K_0(X)$ but do *NOT* generate $D^b(X)$.

Plan for talk

Goal

Provide a *mechanism* for many conjectures about $D^b(X)$ that is more concrete than appealing to homological mirror symmetry.

Key points:

1. Semiorthogonal decompositions (SOD's) of $D^b(X)$ arise from certain paths in $\mathrm{Stab}(X)$, the space of Bridgeland stability conditions on $D^b(X)$
2. These paths are convergent in a partial compactification of $\mathrm{Stab}(X)/\mathbb{G}_a$
3. *Noncommutative MMP* = conjectures about canonical paths on $\mathrm{Stab}(X)/\mathbb{G}_a$ that imply several previous conjectures about $D^b(X)$.

Context

- $\mathscr{C}$ = pre-triangulated dg-category,
- $v : K_0(\mathscr{C}) \twoheadrightarrow \Lambda \cong \mathbb{Z}^n$, called "Mukai vector" homomorphism.

Example (Main)

- $\mathscr{C} = D^b(X)$, X a smooth projective variety,
- v is twisted Chern character map
  $v = (2\pi i)^{\deg/2}\mathrm{ch} : K_0(X) \twoheadrightarrow H^*_{\mathrm{alg}}(X) \subset H^*(X;\mathbb{C})$.

Comparing the definitions

Stability condition:

- $\mathscr{P}_\phi \subset \mathscr{C}$ semistable, $\phi \in \mathbb{R}$
- semiorthogonality for Hom
- every $E \in \mathscr{C}$ has a filtration with $\mathrm{gr}_\phi(E) \in \mathscr{P}_\phi$
- $\mathscr{P}_\phi[1] = \mathscr{P}_{\phi+1}$

Additional data: central charge homomorphism $Z : \Lambda \to \mathbb{C}$ with

- $Z(\mathscr{P}_\phi) \subset \mathbb{R}_{>0} \cdot e^{i\pi\phi}$
- support property

Semiorthogonal decomposition:

- $\mathscr{C}_1, \ldots, \mathscr{C}_n \subset \mathscr{C}$
- semiorthogonality for Hom
- every $E \in \mathscr{C}$ has a filtration with $\mathrm{gr}_i(E) \in \mathscr{C}_i$
- $\mathscr{C}_i[1] = \mathscr{C}_i$

Additional data:

???

Bridgeland stability conditions

Importance of additional data:

Theorem (Bridgeland)

$\mathrm{Stab}(\mathscr{C})$ admits a metric topology such that forgetful map $\mathrm{Stab}(\mathscr{C}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ taking $(\mathscr{P}_\bullet, Z) \mapsto Z$ is a local homeomorphism.

Relevance for this talk:

Paths in $\mathrm{Stab}(\mathscr{C})$ are determined by starting point and a path in $\mathrm{Hom}(\Lambda, \mathbb{C})$.

Key lemma

Let σ_t be a path in $\mathrm{Stab}(\mathscr{C})$ satisfying "quasi-convergence":

1. $\forall E \in \mathscr{C}$, Harder-Narasimhan filtration stabilizes for $t \gg 0$;
2. $\forall$ eventually semistable E,

$$\log Z_t(E) = \alpha_E t + \beta_E + o(1) \text{ for some } \alpha_E, \beta_E \in \mathbb{C};$$

3. if $\Im(\alpha_E) = \Im(\alpha_F)$, then $\alpha_E = \alpha_F$

Lemma (Key Lemma)

$\exists$ a SOD $\mathscr{C} = \langle \mathscr{C}_1, \ldots, \mathscr{C}_n \rangle$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, where $\Im(\alpha_1) < \ldots < \Im(\alpha_n)$ and

$$\mathscr{C}_i \subset \mathscr{C} \text{ is generated by eventually semistable } E \text{ with } \alpha_E = \alpha_i.$$

Furthermore each $\mathscr{C}_i$ **admits a stability condition** whose semistable objects are eventually semistable and $Z_i(E) = e^{\beta_E}$.

Key lemma

Proof idea.

Let $G_j := \mathrm{gr}_j E$ for the eventual HN filtration of E. Then $\phi_t(G_j) \sim \Im(\alpha_{G_j} t + \beta_{G_j})/\pi$ is increasing in j for all $t \gg 0$, so $\Im(\alpha_{G_j})$ is increasing in j. The filtration for the SOD is the coarsening of this filtration that groups terms with the same α. □

Proposition (Partial converse to key lemma)

If $\mathscr{C}$ is smooth and proper, any SOD where all the factors admit stability conditions can be recovered from a quasi-convergent path.

(Collins-Polishchuk gluing)

A proposal

Folklore categorical analogy

(stability condition on $D^b(X)$) $\leftrightarrow$ (ample divisor class on X)

You can not formulate the usual MMP without ample divisors!

Principle

Categorical birational geometry $=$ the study of SOD's of $D^b(X)$ in which every factor admits a stability condition.

$\uparrow$

"polarizable" SOD's

Example: no phantoms

Lemma

If $\mathscr{C}$ is smooth and proper, $\dim(K_0(\mathscr{C}) \otimes \mathbb{Q}) = 1$, and $\mathscr{C}$ admits a stability condition, then $\mathscr{C}$ is generated by a single exceptional object.

So, if SOD is "polarizable" and it looks like it comes from a full exceptional collection on the level of K-theory, then it does.

Example

On the Barlow surface, $D^b(X) = \langle L_1, \ldots, L_{10}, {}^{\perp}\{L_1, \ldots, L_{10}\}\rangle$ can not arise from a quasi-convergent path in $\mathrm{Stab}(X)$.

Plan for the remainder of the talk

1. "Bordification" of $\mathrm{Stab}(\mathscr{C})/\mathbb{G}_a$
2. Formulate the noncommutative minimal model program
3. Discuss consequences

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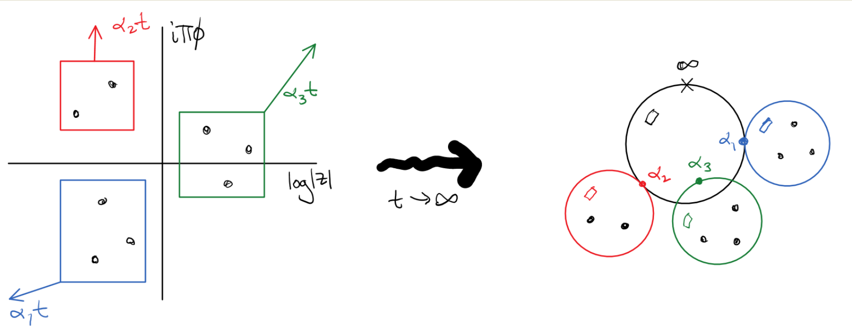
What is going on in key lemma?

Fix E and consider the configuration $\{\log Z_t(\mathrm{gr}_i^{HN}(E))\}_{i=1}^n$ in $\mathbb{C}$:



$(\mathbb{P}^1, dz)$ degenerates to a *multi-scaled line*: a marked genus 0 nodal curve with meromorphic differential (Σ, Ω) with all components isomorphic to $(\mathbb{P}^1, dz)$. (also has a "level structure")

Generalized stability conditions

A generalized stability condition consists of

1. a multi-scaled line $(\Sigma, p_\infty, \Omega)$ with an "order preserving" labeling of terminal components $v_1, \ldots, v_n$
2. an SOD $\mathscr{C} = \langle \mathscr{C}_1, \ldots, \mathscr{C}_n \rangle$
3. elements $\sigma_i \in \mathrm{Stab}(\mathscr{C}_i)/\mathbb{G}_a$ for all i

Regard $\log$ of central charge of σ_i as taking values in the corresponding terminal component of Σ.

(Equivalence relation on generalized stability conditions is slightly non-trivial.)

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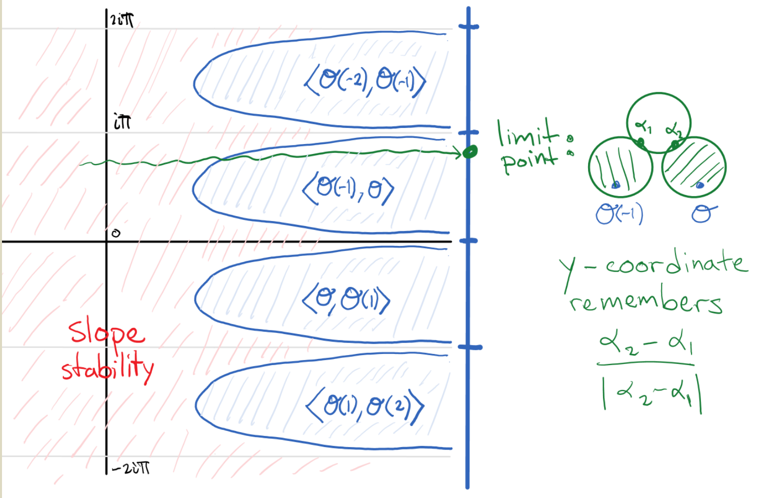
Example of $\mathbb{P}^1$

$\mathrm{Stab}(\mathbb{P}^1)/\mathbb{G}_a \cong \mathbb{C}$. Partially compactified by the blue vertical line at infinity. Green path is quasi-convergent.



The space of generalized stability conditions

In progress (joint with Alekos Robotis):

- There is a Hausdorf space $G\mathrm{Stab}(\mathscr{C})$ containing $\mathrm{Stab}(\mathscr{C})/\mathbb{G}_a$ as a dense open subset.
- There is an s.n.c. compactification $\mathbb{C}^n/\mathbb{G}_a \subset M_n^{ms}$ by n-marked stable multi-scaled lines.
- There are locally defined continuous maps

$$\log Z : U \subset G\mathrm{Stab}(\mathscr{C}) \to \tilde{M}_n^{ms},$$

  where $n = \mathrm{rk}(\Lambda)$ and $\tilde{M}_n^{ms}$ denotes the real oriented blowup of M_n^{ms} along its boundary.
- **Conjecture:** the $\log Z$ maps are local homeomorphisms, making $G\mathrm{Stab}(\mathscr{C})$ a manifold with corners.

The NMMP conjectures

A. To any contraction $\pi : X \to Y$ of a smooth projective X, one can associate a canonical collection of quasi-convergent paths $\sigma_t^{\pi,\psi} \in \mathrm{Stab}(X)/\mathbb{G}_a$, and different generic parameters ψ give mutation equivalent SOD's

B. If $Y \to Y'$ is a further contraction, then for suitable parameters the SOD for $X \to Y'$ refines that for $X \to Y$.

C. If, furthermore, Y is smooth and $R\pi_*(\mathscr{O}_X) = \mathscr{O}_Y$, then for suitable parameters, the SOD for $X \to Y'$ refines the SOD obtained by combining

$$D^b(X) = \langle \ker(\pi_*), \pi^*(D^b(Y)) \rangle$$

with the SOD of $D^b(Y) \cong \pi^*(D^b(Y))$ associated to $Y \to Y'$.

Consequences

Assuming the NMMP conjectures:

Proposition

Given a contraction $X \to Y$ of a smooth projective X with $h^0(K_X) > 0$, $\exists$ an admissible category $\mathscr{M}_{X/Y} \subset D^b(X)$, supported on all of X, such that for any other contraction $X' \to Y$ that is birational to X relative to Y, one has an admissible embedding $\mathscr{M}_{X/Y} \subset D^b(X')$.

Corollary

If $X \dashrightarrow X'$ and $|K_X|$ is baspoint free, then $\exists$ admissible embedding $D^b(X) \hookrightarrow D^b(X')$, which is an equivalence if $|K_{X'}|$ is also basepoint free.

Also: gives canonical categorical resolutions of singularities.

More precise proposal for canonical paths

Ansatz: The central charges for the canonical quasi-convergent paths in $\mathrm{Stab}(X)/\mathbb{G}_a$ should have the form for $E \in D^b(X)$

$$Z_t(E) = \int_X \Phi_t(E),$$

where $\Phi_t(E) \in H^*_{\mathrm{alg}}(X)_{\mathbb{C}}$ is linear in $v(E) \in H^*_{\mathrm{alg}}(X)$ and satisfies a *truncated* quantum differential equation

$$t\frac{\partial \Phi_t(E)}{\partial t} + E_\psi(t)\Phi_t(E).$$

Here $E_\psi(t) \in \mathrm{End}(H^*_{\mathrm{alg}}(X)_{\mathbb{C}})$ depends on a class $\psi = -\omega + iB \in NS(X)_{\mathbb{C}}$ with ω small and relatively ample:

$$(E_\psi(t)\alpha, \beta)_X := \sum_{\substack{d \in N_1(X/Y) \\ c_1(X)\cdot d > \omega \cdot d}} \langle c_1(X), \alpha, \beta \rangle^X_{0,3,d} t^{c_1(X)\cdot d} e^{\psi \cdot d}$$

Relationship to Dubrovin / Gamma conjectures

Iritani defines a "quantum cohomology (QH^*) central charge" $Z_{t,\psi}(E)$, which satisfies the above Ansatz (X Fano).

Proposition

X admits a full exceptional collection (actually, Gamma II holds) if:

- the QH^* central charge lifts to a quasi-convergent path in $\mathrm{Stab}(X)/\mathbb{G}_a$ for generic ψ,
- $Ch : K_0(X) \otimes \mathbb{C} \to H^*(X;\mathbb{C})$ is bijective, and
- QH^* is generically semisimple.

Example

In $\mathrm{Stab}(\mathbb{P}^1)/\mathbb{G}_a \cong \mathbb{C} \cong H^2(\mathbb{P}^1;\mathbb{C})$, the QH^* central charge starts at ψ and moves straight to the right.

Relationship to blowup formula

Decategorification

Apply periodic cyclic homology to SOD of $D^b(X)$

$\Downarrow$

Direct sum decomposition of the Hodge structure on $K_0^{top}(X)$

NMMP implies canonical decompositions of $K^{top}(X)$ up to mutation – roughly an alternative version of the Katzarkov-Kontsevich-Pantev-Yu blowup formula conjecture.