From geometric quantization to noncommutative algebraic geometry

Noncommutative Shapes

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0. Motivation

Generalized complex structure $\begin{cases} \bullet\ \mathbb{I} : TM \oplus T^*M \circlearrowleft \quad \mathbb{I}^2 = -1 \\ \bullet\ \text{integrable for Courant bracket} \end{cases}$

Symplectic		Complex
$\mathbb{I}_A = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$		$\mathbb{I}_B = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}$
$H^2(M, \mathbb{C})$	Deformations	$H^0(\Lambda^2 T) \oplus H^1(T) \oplus H^2(\mathcal{O})$
A-model (loc. to pseudohol. maps)	2d Sigma model	B-model (loc. to constant maps)
Lagrangians* $D^\pi(Fuk(M))$	Category of Boundary conditions	Holomorphic sheaves $D^b(\mathrm{Coh}\, M)$

0. Motivation

Holomorphic Poisson deformations

$\sigma \in H^0(\Lambda^2 T)$ holomorphic bivector $\sigma = -\frac{1}{4}(P + iQ)$

$$\mathbb{I}_\sigma = \begin{pmatrix} I & Q \\ 0 & -I^* \end{pmatrix} \iff \underbrace{\bar{\partial}\sigma = 0}_{\text{Holomorphic}} \quad \text{and} \quad \underbrace{[\sigma, \sigma] = 0}_{\text{Poisson}}$$

is integrable

Question: Can generalized complex geometry give a geometric approach to noncommutative algebraic geometry?

Some Algebras Associated to Automorphisms of Elliptic Curves

M. ARTIN, J. TATE and M. VAN DEN BERGH

1. Introduction

The main object of this paper is to relate a certain type of graded algebra, namely the regular algebras of dimension 3, to automorphisms of elliptic curves. Some of the results were announced in [V]. A graded algebra A is called *regular* if it has finite global dimension, polynomial growth, and is Gorenstein. The precise definitions are reviewed in Section 2. As was shown in [A-S], there are two basic possibilities for a regular algebra A of (global) dimension 3 which is generated in degree 1. Either A can be presented by 3 generators and 3 quadratic relations, or else by 2 generators and 2 cubic relations. Throughout this paper, A will denote an algebra so presented, over a ground field k. The number of generators will be denoted by r, and the degrees of the defining relations by s. Thus the possible values are

$$(1.1) \qquad (r,s) = \begin{cases} (3,2) \\ (2,3), \end{cases} \quad \text{and } r+s=5.$$

In ([A-S], (1.5)) it is shown that if A is regular, then there are choices of generators (x_i) and relations $(f_i = 0)$, $1 \leq i \leq r$, such that if we

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Inventiones mathematicae

Modules over regular algebras of dimension 3

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1 Introduction

Let k be a field. In a previous paper [ATV] (see also [OF]) some graded k-algebras A, regular algebras of dimension 3, were constructed from certain automorphisms σ of elliptic curves or of more general one-dimensional schemes E with arithmetic genus 1, which are embedded as cubics in $\mathbb{P}^2$ or as divisors of bidegree (2, 2) in $\mathbb{P}^1 \times \mathbb{P}^1$. In this correspondence, the points of the scheme E were shown to parametrize certain A-modules called *point modules.* A point module N is a graded right A-module with these properties:

(1.1) (i) $N_0 = k$,
(ii) N_0 generates N, and
(iii) $\dim_k N_i = 1$ for all $n \geqq 0$.

The structure of these point modules is related in a nice way to the geometry of the scheme E and its automorphism σ. For example, if $N = N_p$ is the module corresponding to a point p of E, then the normalized shift N^+, defined by

$$(1.2) \qquad N_i^+ = \begin{cases} N_{i+1} & \text{if } i \geqq 0 \\ 0 & \text{if } i < 0 \,, \end{cases}$$

Brane quantization of toric Poisson varieties

Francis Bischoff* Marco Gualtieri†

Abstract

In this paper we propose a noncommutative generalization of the relationship between compact Kähler manifolds and complex projective algebraic varieties. Beginning with a prequantized Kähler structure, we use a holomorphic Poisson tensor to deform the underlying complex structure into a generalized complex structure, such that the prequantum line bundle and its tensor powers deform to a sequence of generalized complex branes. Taking homomorphisms between the resulting branes, we obtain a noncommutative deformation of the homogeneous coordinate ring. As a proof of concept, this is implemented for all compact toric Kähler manifolds equipped with an R-matrix holomorphic Poisson structure, resulting in what could be called noncommutative toric varieties.

To define the homomorphisms between generalized complex branes, we propose a method which involves lifting each pair of generalized complex branes to a single coisotropic A-brane in the real symplectic groupoid of the underlying Poisson structure, and compute morphisms in the A-model between the Lagrangian identity bisection and the lifted coisotropic brane. This is done with the use of a multiplicative holomorphic Lagrangian polarization of the groupoid.

Contents

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Project Summary

Kähler: (M, ω) + Polarization I

B-brane pair $\mathcal{O}, \mathcal{L}$

Quantization $\mathrm{Hom}_B(\mathcal{O}, \mathcal{L})$

B-Endofunctor $\mathcal{E} \mapsto \mathcal{L} \otimes \mathcal{E}$

Graded algebra $A^\bullet = \bigoplus_{k \geq 0} \mathrm{Hom}_B(\mathcal{O}, \mathcal{L}^k)$

DEFORM

$I \to \mathbb{I}$

Generalized complex deformation: (M, ω) + Pol. $\mathbb{I}$

GC brane pair $\mathcal{O}, \mathcal{L}$

Quantization $\boxed{\mathrm{Hom}_{\mathbb{I}}(\mathcal{O}, \mathcal{L})}$ ← undefined

Endofunctor $\mathcal{E} \overset{\phi}{\longmapsto} \mathcal{L} \otimes \varphi^* \mathcal{E}$

Graded algebra $\boxed{A^\bullet = \bigoplus_{k \geq 0} \mathrm{Hom}_{\mathbb{I}}(\mathcal{O}, \phi^k(\mathcal{O}))}$

B	$\mathbb{I}$
A	A

COTANGENT LIFT

INTEGRATION TO GROUPOID

Cotangent: (T^*M, ω_0)

A-brane pair $\begin{cases} \mathcal{E} & \text{Lagrangian} = 0_M \\ \mathcal{B} & \text{Space-filling} \end{cases}$

Quantization $\mathrm{Hom}_A(\mathcal{E}, \mathcal{B})$

Tensor product on A-branes $T^*M \times T^*M \overset{\oplus}{\dashrightarrow} T^*M$

Graded algebra $A^\bullet = \bigoplus_{k \geq 0} \mathrm{Hom}_A(\mathcal{E}, \mathcal{B}^k)$

DEFORM

groupoid

Symplectic groupoid of $\mathbb{I}$: (G, ω_0)

A-brane pair $\begin{cases} \mathcal{E} & \text{Lagrangian} = 0_M \\ \mathcal{B}_\phi & \text{Space-filling} \end{cases}$

Quantization $\boxed{\mathrm{Hom}_A(\mathcal{E}, \mathcal{B}_\phi)}$ ← propose def^n

Tensor product on A-branes $G \times G \overset{m}{\dashrightarrow} G$

Graded algebra (Noncommutative) $\boxed{A^\bullet = \bigoplus_{k \geq 0} \mathrm{Hom}_A(\mathcal{E}, \mathcal{B}_\phi^k)}$

1 Geometric quantization of Kähler manifolds: B-model approach

- (M, ω) symplectic with periods $\in \mathbb{Z}$
- (L, ∇) prequantization $\quad \operatorname{curv} \nabla = -2\pi i\, \omega$
- complex polarization I, $\quad \omega I + I^* \omega = 0$
- quantization $\quad H^0(M, L) = \ker \nabla^{0,1}$

$$\boxed{\begin{gathered} \mathcal{A} = \bigoplus_{k \geq 0} \operatorname{Hom}_B(\mathcal{O}, \phi^k(\mathcal{O})) \\ \phi : \operatorname{Coh} M \to \operatorname{Coh} M \\ \mathcal{E} \longmapsto L \otimes \mathcal{E} \end{gathered}}$$

$$\|$$

Scaling family

$\omega, \quad 2\omega, \quad 3\omega, \quad \ldots$

$$\xrightarrow{\text{quantization}}$$

Graded algebra

$$\mathcal{A} = \bigoplus_{k \geq 0} H^0(M, L^{\otimes k})$$

Kodaira embedding theorem: this captures polarization: algebraic variety.

2. Geometric quantization of Kähler manifolds: A-model approach

Encode Kähler geometry (M, I, ω) as a pair of A-branes in (T^*M, ω_0)

- Lagrangian brane: $\mathcal{L} = (\underline{O}_M, \underline{\mathbb{C}}_M, d)$ zero section
- Space-filling brane: $\mathcal{B} = (\mathcal{U}, \hat{\nabla}) = (\pi^* L, \pi^*\nabla - 2\pi i \,\mathrm{Re}\,\alpha_0)$ $\quad d\alpha_0 = \Omega_0$

Im

hol. symp. form on T^*M

Canonical holomorphic symplectic form on T^*M

$$\hat{F} + i\omega_0 = \Omega_0 + \pi^*\omega$$ Twisted holomorphic cotangent bundle

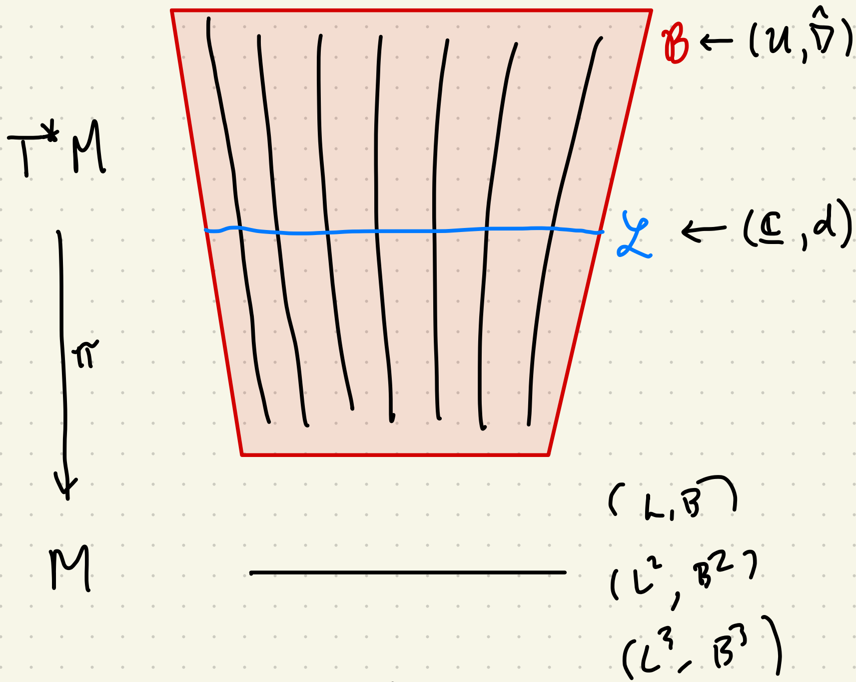
Def^n: $(\mathcal{U}, \hat{\nabla})$ space-filling brane in (X, ω_0) when $\hat{F} + i\omega_0$ is a holomorphic symplectic form.

Intersection recovers Kähler structure: $\quad (\mathcal{U}, \hat{\nabla})|_{\mathcal{L}} = (L, \nabla)$

Holomorphic prequantization of $\mathcal{B}$: $\quad (\mathcal{U}, D) = (\mathcal{U}, \hat{\nabla} + 2\pi\,\mathrm{Im}\,\alpha_0)$ $\quad \mathrm{curv}\, D = -2\pi i(\hat{F} + i\omega_0)$

Holomorphic Lagrangian Polarization: cotangent fibers

2. Geometric quantization of Kähler manifolds : A-model approach



Polarization: fibers of π are hol. Lagrangian

Quantization

$\mathrm{Hom}_A(\mathcal{L}, \mathcal{B})$ = holomorphic flat sections along Polariz $\cap\, \mathcal{L}$

"Fukaya hom"

$= H^0(M, L) = \mathrm{Hom}_B(\mathcal{O}, L)$

Algebra

$T^*M \times_M T^*M \xrightarrow{\oplus} T^*M$ Symplectic groupoid

Convolution of branes $\begin{cases} L * L = L \\ \mathcal{B} * \mathcal{B} = \mathcal{B}^2 \end{cases}$

$$\boxed{\begin{aligned} \mathcal{A} &= \bigoplus_{k \geq 0} \mathrm{Hom}_A(\mathcal{L}, \mathcal{B}^k) \\ &= \bigoplus_{k \geq 0} \mathrm{Hom}_B(\mathcal{O}, L^k) \end{aligned}}$$

2. Geometric quantization of Kähler manifolds : A-model approach

Branes And Quantization

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The problem of quantizing a symplectic manifold (M, ω) can be formulated in terms of the A-model of a complexification of M. This leads to an interesting new perspective on quantization. From this point of view, the Hilbert space obtained by quantization of (M, ω) is the space of $(\mathcal{B}_{cc}, \mathcal{B}')$ strings, where $\mathcal{B}_{cc}$ and $\mathcal{B}'$ are two A-branes; $\mathcal{B}'$ is an ordinary Lagrangian A-brane, and $\mathcal{B}_{cc}$ is a space-filling coisotropic A-brane. $\mathcal{B}'$ is supported on M, and the choice of ω is encoded in the choice of $\mathcal{B}_{cc}$. As an example, we describe from this point of view the representations of the group $SL(2,\mathbb{R})$. Another application is to Chern-Simons gauge theory.

Probing Quantization Via Branes

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Abstract

We re-examine quantization via branes with the goal of understanding its relation to geometric quantization. If a symplectic manifold M can be quantized in geometric quantization using a polarization $\mathcal{P}$, and in brane quantization using a complexification Y, then the two quantizations agree if $\mathcal{P}$ can be analytically continued to a holomorphic polarization of Y. We also show, roughly, that the automorphism group of M that is realized as a group of symmetries in brane quantization of M is the group of symplectomorphisms of M that can be analytically continued to holomorphic symplectomorphisms of Y. We describe from the point of view of brane quantization several examples in which geometric quantization with different polarizations gives equivalent results.

3. Generalized complex deformation

$\sigma = \frac{-1}{4}(P + iQ) \in H^0(\Lambda^2 T)$ holomorphic Poisson

$$\mathbb{I} = \begin{pmatrix} I & Q \\ 0 & -I^* \end{pmatrix}$$ generalized complex structure on M

Def^n: A brane on $(M, \mathbb{I})$ is (L, ∇) unitary line bundle s.t. $\mathbb{I}\,\Gamma_{F_\nabla} = \Gamma_{F_\nabla}$, i.e.

$$F_\nabla I + I^* F_\nabla + F_\nabla Q F_\nabla = 0 \qquad \left(\text{when } Q = 0 \text{ recover } F_\nabla \in \Omega^{1,1}\right)$$

Thm (-): If the prequantum line bundle is a Poisson module ($\exists$ invariant lift of σ)
Then it deforms to a brane for $\mathbb{I}$.

Metric h on L $\Rightarrow$ $\hat{Q}$ - Hamiltonian on tot(L)
$\Rightarrow$ Q - Poisson vector field on M $\Rightarrow$ φ_t flow
$\Rightarrow$ $\overline{\nabla} = \int_0^1 \varphi_t^* \nabla \; dt$ Brane for $\mathbb{I}$ $= B_1$

Trivial brane $(\mathbb{C}_M, d) = B_0$

3. Generalized complex deformation

$$\begin{cases} \text{Trivial brane} & B_0 = (\mathbb{C}_M, d) \\ \text{Deformed prequantum brane} & B_1 = (L, \bar{\nabla}) \end{cases}$$

Quantization: $\boxed{\mathrm{Hom}_{\mathbb{I}}(B_0, B_1)}$

Algebra: Iterate the autoequivalence $\quad B \xmapsto{T} (L, \bar{\nabla}) \otimes \varphi_1^*(B)$

$$B_0, \; B_1 = T(B_0), \; B_2 = T^2(B_0), \ldots \qquad \boxed{\mathcal{A} = \bigoplus_{k \geq 0} \mathrm{Hom}_{\mathbb{I}}(B_0, B_k)}$$

Problem: Morphisms between $\mathbb{I}$-branes are not defined.

Deformed B-model approach is stuck.

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DEFORM

$\mathrm{I} \to \mathrm{II}$

Generalized complex deformation: (M, ω) + Pol. II

GC brane pair $\mathcal{O}, \mathcal{L}$

Quantization $\boxed{\mathrm{Hom}_{\mathrm{II}}(\mathcal{O}, \mathcal{L})}$ ← undefined

Endofunctor $\mathcal{E} \overset{\phi}{\longmapsto} \mathcal{L} \otimes \varphi^* \mathcal{E}$

Graded algebra $\boxed{A^\bullet = \bigoplus_{k \geq 0} \mathrm{Hom}_{\mathrm{II}}(\mathcal{O}, \phi^k(\mathcal{O}))}$

COTANGENT LIFT ↓ $\cong$

$$\begin{array}{c|c} B & \mathrm{II} \\ \hline A & A \end{array}$$

↓ INTEGRATION TO GROUPOID

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Quantization $\boxed{\mathrm{Hom}_A(\mathcal{E}, \mathcal{B}_\phi)}$ ← propose def^n

Tensor product on A-branes $G \times G \overset{m}{\dashrightarrow} G$

Graded algebra (Noncommutative) $\boxed{A^\bullet = \bigoplus_{k \geq 0} \mathrm{Hom}_A(\mathcal{E}, \mathcal{B}_\phi^k)}$

3. Generalized complex deformation: A-model approach

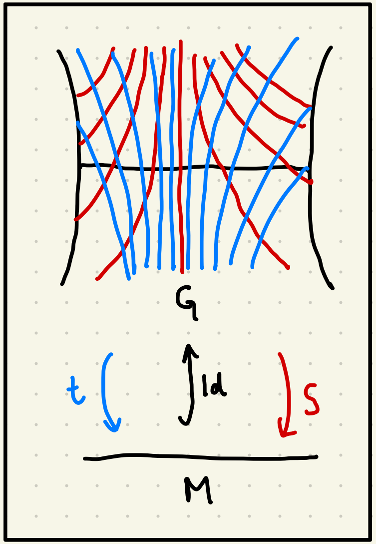
Any GC structure $\begin{pmatrix} A & Q \\ \tau & -A^* \end{pmatrix}$ has underlying real Poisson structure Q

Q may be "integrated" to a (local) symplectic groupoid $G \overset{s}{\underset{t}{\rightrightarrows}} M$

- $(G, s, t, m, \mathrm{id})$ Lie groupoid
  (Hamiltonian paths up to Ham. homotopy)

- ω_0 Symplectic form s.t. multiplication is a Lagrangian relation in $\bar{G} \times \bar{G} \times G$

$$\begin{tikzcd} & G \times_M G \arrow[dl] \arrow[dr, "m"] & \\ G \times G & & G \end{tikzcd}$$



- for $Q = 0$, $G = T^*M$ with $m =$ fibrewise addition

3. Generalized complex deformation: A-model approach

Key properties:

① $\mathcal{E} = \mathrm{Id}(M) \subset G$ is Lagrangian

② $(t,s) : (G, \omega_0) \longrightarrow (M \times M, \mathbb{I} \times \mathbb{I}^-)$ generalized holomorphic

B_i, B_j $\mathbb{I}$-branes $\Rightarrow (t,s)^{-1}(B_j \times B_i) = B_{ji}$ A-brane in (G, ω_0)

Quantization: $\mathrm{Hom}_{\mathbb{I}}(B_0, B_1) := \mathrm{Hom}_A(\mathcal{E}, B_{10})$

③ Convolution product on branes $\quad G \times G \overset{m}{\dashrightarrow} G$

$\mathrm{EFuk}(G, \omega_0)$ Monoidal $\quad B' \times B \longmapsto B' * B$

Graded algebra: $\mathcal{A} = \bigoplus_{k \geq 0} \mathrm{Hom}_A(\mathcal{E}, B_{10}^{*k})$

4. A-model Homs Lagrangian $\to$ Space-filling

$$\text{Space-filling } \mathcal{B} \longrightarrow (G, \omega_0) \longleftarrow \mathcal{E} \text{ Lagrangian}$$

Proposed definition: • $\mathcal{B} = (U, \nabla)$ s.t. $F_\nabla + i\omega_0 = \Omega$ holomorphic symplectic

choose $\left\{ \begin{array}{l} \bullet \text{ Holomorphic prequantization } (U, D) \quad \mathrm{curv}(D) = \Omega \\ \bullet \text{ Holomorphic Lagrangian foliation } \mathcal{F} \text{ of } (G, \Omega) \end{array} \right.$

Quantization:

$$\boxed{\mathrm{Hom}(\mathcal{E}, \mathcal{B}) = \bigoplus_{\substack{\text{BS Leaves} \\ \text{intersecting } \mathcal{E}}} H^0(U)^{D|_{\mathcal{F}}}}$$

4. A-model Homs Lagrangian $\longrightarrow$ Space-filling

$$\text{Space-filling } \mathcal{B} \longrightarrow (G,\omega_0) \longleftarrow \mathcal{E} \text{ Lagrangian}$$

Algebra:

- take iterated convolution $\mathcal{B}, \mathcal{B}^{*2}, \mathcal{B}^{*3}, \dots$
- multiplicative prequantization
- multiplicative polarization

multiplicative cocycle

$$(\mathcal{U}_i \boxtimes \mathcal{U}_j, D_i \boxtimes D_j) \xrightarrow{\Theta} (\mathcal{U}_{i+j}, D_{i+j})$$

$$\begin{tikzcd}
(\mathcal{U}_i \boxtimes \mathcal{U}_j, D_i \boxtimes D_j) \arrow[d, squiggly] & G \times_M G \arrow[dl, "\pi"'] \arrow[dr, "m"] & (\mathcal{U}_{i+j}, D_{i+j}) \arrow[d, squiggly] \\
G \times G & & G
\end{tikzcd}$$

$$\boxed{\begin{aligned} \mathcal{A} = &\bigoplus_{k\geq 0} \mathrm{Hom}_A(\mathcal{E}, \mathcal{B}^{*k}) \\ &\bigoplus_{k\geq 0} \Big(\bigoplus_{L\in BS\cap\mathcal{E}} H^0(\mathcal{U}_k|_L) \Big) \end{aligned}}$$

5. Implementation for Toric R-matrix Poisson varieties

① (M, I, ω) Toric Kähler $\qquad \begin{array}{ccc} \mathbb{T}_{\mathbb{C}} & \xrightarrow{\rho} & \mathrm{Aut}(M,I) \\ \cup & & \cup \\ \mathbb{T} & \longrightarrow & \mathrm{Aut}(M,I,\omega) \end{array}$

Prequantization (L, ∇) + linearization $\hat{\rho}$

② lift $\mathbb{T}_{\mathbb{C}}$ action to T^*M $\Rightarrow$ Hamiltonian $J : T^*M \longrightarrow \mathfrak{t}_{\mathbb{C}}^*$ Momentum

③ R-matrix $C \in \Lambda^2 \mathfrak{t}_{\mathbb{C}}$ (Invariant Poisson str on $\mathbb{T}_{\mathbb{C}}$)

$\rho_* C = \sigma_C$ Holomorphic Poisson on M

$\hat{\rho}_* C = \hat{\sigma}_C$ Poisson module str. on L

5. Implementation for Toric R-matrix Poisson varieties

Symplectic groupoid of (M, σ_c) is $T^*M \underset{t}{\overset{s}{\rightrightarrows}} M$ with

$z \in T^*M$

$$t(z) = e^{\frac{1}{2}cJ(z)}\pi(z) \qquad s(z) = e^{-\frac{1}{2}cJ(z)}\pi(z)$$

$$m(x,y) = e^{-\frac{1}{2}cJ(y)}x + e^{\frac{1}{2}cJ(x)}y$$

(U, D) is as in Kähler case, but cocycle is nontrivial:

$$\boxed{H}_c(x,y)(f \otimes g) = e^{i\pi c(J(x),J(y))}\left(e^{-\frac{1}{2}cJ(y)}f\right) \otimes \left(e^{\frac{1}{2}cJ(x)}g\right)$$

5. Implementation for Toric R-matrix Poisson varieties

④ Bohr-Sommerfeld Leaves for $\mathcal{B}^{*K} \subset G$ $\overset{1:1}{\longleftrightarrow}$ Integer lattice points in $\mathfrak{t}^*_{\mathbb{R}}$

Bohr-Sommerfeld Leaves intersecting $\mathcal{E}$ $\overset{1:1}{\longleftrightarrow}$ Integer lattice points in Delzant polytope for $k\omega$.

Theorem 3.6. *The algebra $\mathcal{A}_C$ and the homogeneous coordinate ring of the toric variety have canonically isomorphic underlying graded vector spaces*

$$\mathcal{A}_C \cong \bigoplus_{n \geq 0} H^0(M, L^{\otimes n}).$$

Under this identification the product of two homogeneous sections f and g, with respective $\mathbb{T}_{\mathbb{C}}$-weights $w_1, w_2 \in \mathfrak{t}^*_{\mathbb{C}}$, is given by

$$f * g = e^{\frac{i}{4\pi} C(w_1, w_2)} f \otimes g,$$

resulting in the commutation relations

$$f * g = e^{\frac{i}{2\pi} C(w_1, w_2)} g * f.$$

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