noncommutative Shapes

a conference in honour of Michel Van den Bergh

University of Antwerp September 12-16, 2022

Mirror Symmetry for Adjoint Orbits

Mirror Symmetry for Adjoint Orbits

Elizabeth Terezinha Gasparim

16 September 2022

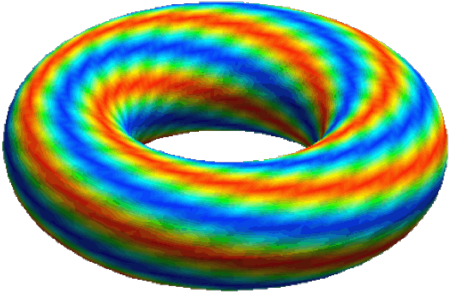
Geometric Mirror Symmetry

$$
\begin{array}{ccccccccc}
 & & & & h^{n,n} & & & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & & \vdots & & & \ddots & \\
h^{n,0} & & & \cdots & \overset{\curvearrowleft}{\mathit{Serre}} & \cdots & & & h^{0,n} \quad \updownarrow \mathit{Hodge}\star \\
 & \ddots & & & \vdots & & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & & & h^{0,0} & & & & \\
 & & & & \underset{\mathit{Conjugation}}{\longleftrightarrow} & & & &
\end{array}
$$

Geometric Mirror Symmetry

$$
\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & \\
 & & & \vdots & & & \ddots \\
h^{n,0} & & \cdots & h^{m,m} & \cdots & & h^{0,n} \quad \updownarrow \mathit{Hodge}\star \\
 & \ddots & & \vdots & & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & & \\
 & & & \overset{\longleftrightarrow}{\mathit{Conjugation}} & & &
\end{array}
$$

Example 1: Smooth Elliptic Curve



$$h^{1,1} = 1$$

$$h^{1,0} = 1 \qquad\qquad h^{0,1} = 1$$

$$h^{0,0} = 1$$

Example 2: Hodge Diamond of a K3 Surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

A Calabi Yau threefold and its mirror

Calabi Yau X	$\hat{X}$ = Mirror of X
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$	$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Candelas, de la Ossa, Green, Parkes (1991)

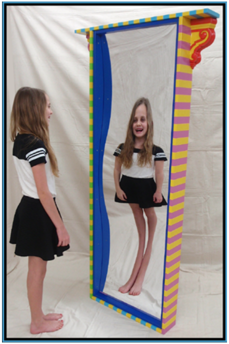
Homological Mirror Symmetry (Kontsevich ICM 1994)

$$\{\textit{symplectic geometry}\} \longleftrightarrow \{\textit{algebraic geometry}\}$$

Mirror 1

Different Mirrors

Mirror 2



Bad Mirrors

Mirrors 3

Vampire

Homological Mirror Symmetry Conjecture I

For every LG model (Y, f) there exists a (subvariety of) a projective variety X satisfying the categorical equivalence

$$Fuk(Y, f) = D^b(Coh\, X)$$

Conjectured by Kontsevich at ICM 1994.

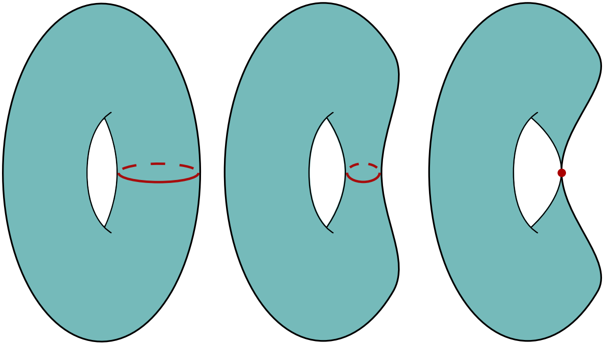
Homological Mirror Match

A	B
Symplectic	Kähler
$Floer(L_i, L_j)$	$Hom(F_i, F_j)$
Lagrangian	Variety
$Fuk(Y, f)$	$D^b Coh(X)$
Thimble	Sheaf

Landau–Ginzburg models

For us a Landau–Ginzburg model is a symplectic or a complex manifold Y together with a complex function $f\colon Y \to \mathbb{C}$ called the superpotential.

Landau–Ginzburg models



Symplectic families with 1 dimensional parameter space and only Morse type singularities.

LG models - examples

Especially well behaved LG models: Lefschetz fibrations!

Topological Lefschetz fibration

Let Y be a complex variety. A smooth function $f\colon Y \to \mathbb{C}$ (or $f\colon Y \to \mathbb{P}^1$) is a Topological Lefschetz fibration if:

- f has finitely many critical points of (holomorphic) Morse type so that around each critical point

$$f(z_0, \ldots, z_n) \simeq z_0^2 + \cdots + z_n^2.$$

- $f|_{Y - \{singular\ fibres\}}$ is locally trivial.

Symplectic Lefschetz fibrations

A a topological Lefschetz fibration $f\colon Y \to \mathbb{C}$ on a symplectic manifold (Y, ω) is a symplectic Letschetz fibration if:

- for every regular value $p \in \mathbb{C}$, the level Y_p is a symplectic submanifold of Y, and
- for each singular point Q_i the symplectic form ω_{Q_i} is non degenerate over the tangent cone of Y_{Q_i} at Q_i.

Examples from algebraic geometry

Donaldson proved that every symplectic 4 manifold has the structure of Lefschetz pencil.

Blow-up the base locus of the pencil to obtain a TLF.

TLFs from pencils

Modify a Lefschetz pencil by blowing up the base locus transforming it to a TLF.

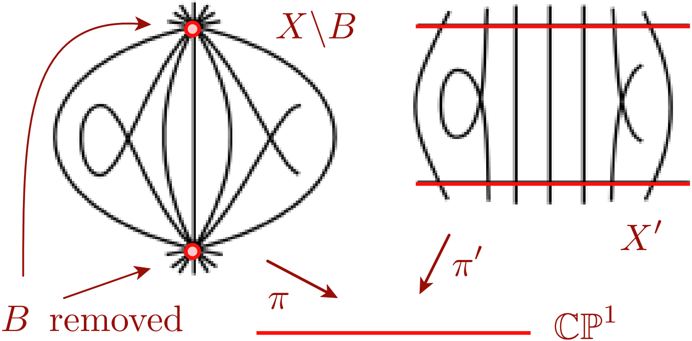


Figure: Pencil to fibration.

Symplectic Lefschetz fibrations via Lie theory

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h}$. Given the Hermitian form $\mathcal{H}$ on $\mathfrak{g}$, define the symplectic form on $\mathfrak{g}$ by

$$\omega(X, Y) = \mathrm{im}\,\mathcal{H}\left(X_1, X_2\right).$$

For $H_0 \in \mathfrak{h}$ we consider the adjoint orbit:

$$Y = \mathcal{O}\left(H_0\right) = \mathrm{Ad}(G) \cdot H_0 = \{\mathrm{Ad}(g) \cdot H_0 \in \mathfrak{g} : g \in G\},$$

together with the symplectic form ω.

Theorem (G., Grama, San Martin)

Given $H_0 \in \mathfrak{h}$ *and* $H \in \mathfrak{h}_{\mathbb{R}}$ *with* H *a regular element. The "height function"* $f_H : (\mathcal{O}(H_0), \omega) \to \mathbb{C}$ *defined by*

$$\boxed{f_H(x) = \langle H, x \rangle \qquad x \in \mathcal{O}(H_0)}$$

has a finite number of isolated singularities and defines a symplectic Lefschetz fibration.

My favorite part of the proof

Lemma

x is a critical point for f_H if and only if $x \in \mathcal{O}(H_0) \cap \mathfrak{h} = \mathcal{W} \cdot H_0$, where $\mathcal{W}$ is the Weyl group.

Orbit and cotangent bundles of flags

Let $K \subset G$ compact, and $\mathbb{F}_0 := \mathrm{Ad}(K) \cdot H_0$ the flag manifold.

Orbit and cotangent bundles of flags

Let $K \subset G$ compact, and $\mathbb{F}_0 := \mathrm{Ad}(K) \cdot H_0$ the flag manifold.

Theorem (G., Grama, San Martin)

There is a $\mathbb{C}^\infty$ isomorphism $\iota \colon \mathrm{Ad}(G) \cdot H_0 \to T^*\mathbb{F}_0$ such that

1. *ι is equivariant with respect to the action of K.*
2. *The pullback of the canonical symplectic form on $T^*\mathbb{F}_0$ by ι is the (real) Kirillov–Kostant–Souriau form on the orbit.*

Corollary

The homology of a regular fibre coincides with the homology of $\mathbb{F}_0 \setminus \mathcal{W} \cdot H_0$. In particular the middle Betti number is $k-1$ where k is the number of singularities of the fibration (equal the number of elements in $\mathcal{W} \cdot H_0$).

Katzarkov–Kontsevich–Pantev conjecture

Notation: $\mathrm{LG}(n)$ denotes the Landau–Ginzburg model defined over the minimal semisimple adjoint orbit of $\mathfrak{sl}(n, \mathbb{C})$.

Theorem (Ballico, G., Rubilar, San Martin)

$\mathrm{LG}(n)$ *satisfies the KKP conjecture.*

Notation: $\mathrm{LG}(n)$ denotes the Landau–Ginzburg model defined over the minimal semisimple adjoint orbit of $\mathfrak{sl}(n, \mathbb{C})$.

Theorem (Ballico, G., Rubilar, San Martin)

$\mathrm{LG}(n)$ *satisfies the KKP conjecture.*

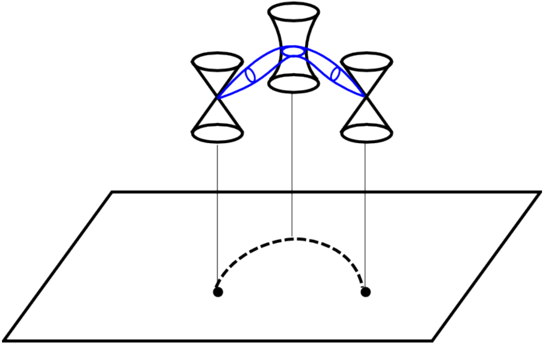
Theorem (Lunts, Przyjalkowski)

The KKP conjecture fails in dimension 2.

Fukaya category

A category whose Lagrangian thimbles associated to the vanishing cycles.

Vanishing cycle and thimble



Vanishing cycle in Physics

Definition

The category of vanishing cycles $Lag(Y, w, \gamma_i)$ is an A_∞-category which objects $L_0, \ldots, L_r$ corresponding to the thimbles. The morphisms between the objects are given by

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j; R) = R^{[L_i \cap L_j]} & \textit{if} \quad i < j \\ R \cdot id & \textit{if} \quad i = j \\ 0 & \textit{if} \quad i > j \end{cases}$$

The differential m_1, composition m_2 and higher order products M_k are defined in terms of Lagrangian Floer homology.

Dehn twist

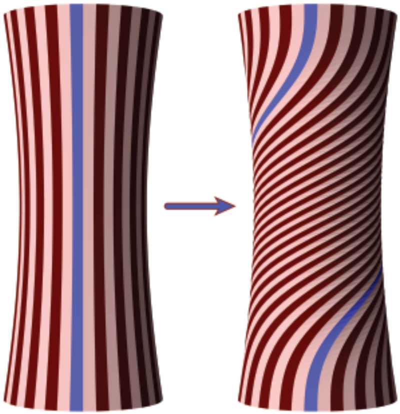


Illustration of generalized Dehn twist

Define the Hamiltonian action

$$\begin{aligned} \mu\colon\ T^*\mathbb{S}^n &\to \mathbb{R} \\ (q,p) &\mapsto |p|, \end{aligned}$$

which is smooth outside the zero section, and use a bump function $r\colon \mathbb{R} \to \mathbb{R}$ to obtain the smooth Dehn twist

$$\tau(x) := e^{i2\pi r'(\mu)} \cdot x.$$

The Fibre $F_0 := T^*_{(0,0,1)}\mathbb{S}^2$

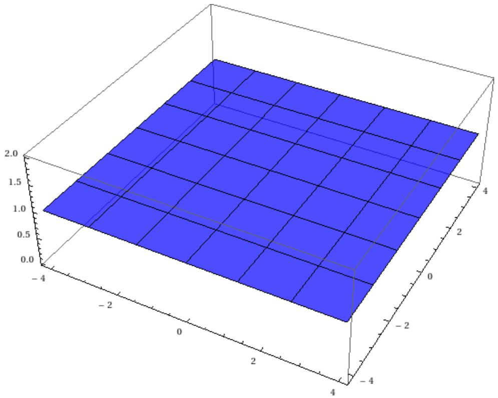


Figure: The Lagrangian $F_0 := T^*_{(0,0,1)}\mathbb{S}^2$.

The Hamiltonian deformation of F_0

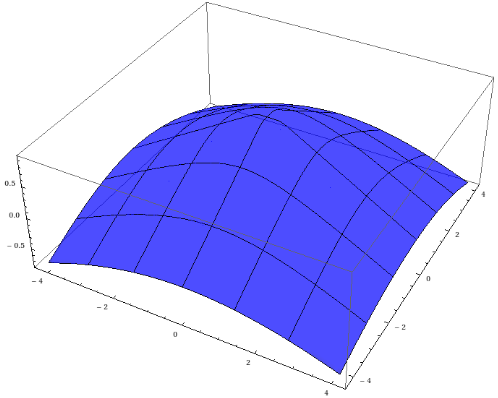


Figure: The deformation $\varphi_t(F_0)$ of the Lagrangian F_0.

The Dehn Twist F_1 of F_0

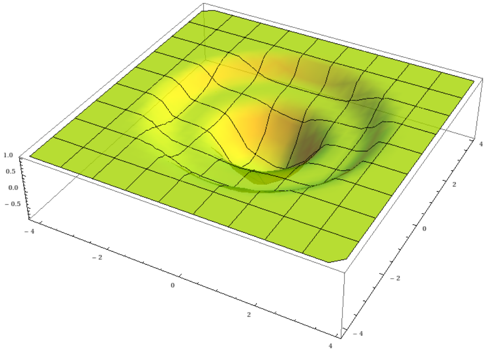


Figure: The Dehn twist $F_1 := \tau(F_0)$ obtained using the map $r(t) := \frac{-\sqrt{t^2+\varphi(t)}+t}{2}$.

Hamiltonian deformation of F_1

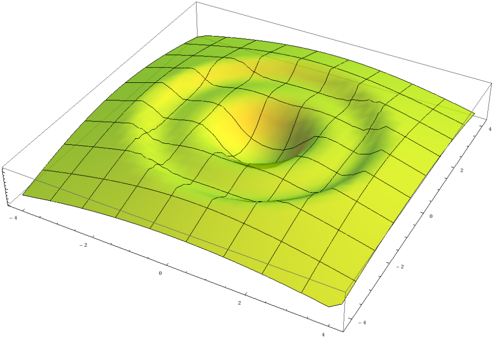


Figure: The deformation of the Dehn twist: $\varphi_t\,(F_1)$.

Floer homology $HF(F_1, F_0)$

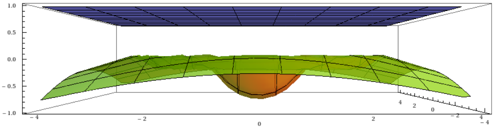


Figure: Intersecting F_0 with $\varphi_t\,(F_1)$.

Floer homology $HF(F_0, F_1)$

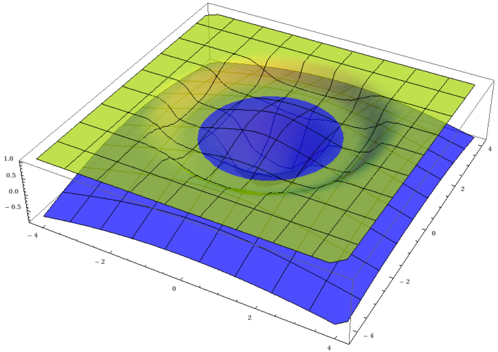


Figure: Intersecting F_1 with $\varphi_t\,(F_0)$.

HMS for the adjoint orbit $\mathcal{O}_2$ of $\mathfrak{sl}(2,\mathbb{C})$

Choose in $\mathfrak{sl}(2,\mathbb{C})$ the elements

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Hence $\mathcal{O}_2$ is the set of matrices in $\mathfrak{sl}(2,\mathbb{C})$ with eigenvalues ± 1.

HMS for the adjoint orbit $\mathcal{O}_2$ of $\mathfrak{sl}(2,\mathbb{C})$

Choose in $\mathfrak{sl}(2,\mathbb{C})$ the elements

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Hence $\mathcal{O}_2$ is the set of matrices in $\mathfrak{sl}(2,\mathbb{C})$ with eigenvalues ± 1.
- $\mathcal{O}_2$ forms a submanifold of $\mathfrak{sl}(2,\mathbb{C})$ of real dimension 4 (a complex surface).

HMS for the adjoint orbit $\mathcal{O}_2$ of $\mathfrak{sl}(2, \mathbb{C})$

Choose in $\mathfrak{sl}(2, \mathbb{C})$ the elements

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Hence $\mathcal{O}_2$ is the set of matrices in $\mathfrak{sl}(2, \mathbb{C})$ with eigenvalues ± 1.
- $\mathcal{O}_2$ forms a submanifold of $\mathfrak{sl}(2, \mathbb{C})$ of real dimension 4 (a complex surface).
- In this case the Weyl group is $\mathcal{W} = \{\pm 1\}$.

HMS for the adjoint orbit $\mathcal{O}_2$ of $\mathfrak{sl}(2,\mathbb{C})$

Choose in $\mathfrak{sl}(2,\mathbb{C})$ the elements

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Hence $\mathcal{O}_2$ is the set of matrices in $\mathfrak{sl}(2,\mathbb{C})$ with eigenvalues ± 1.
- $\mathcal{O}_2$ forms a submanifold of $\mathfrak{sl}(2,\mathbb{C})$ of real dimension 4 (a complex surface).
- In this case the Weyl group is $\mathcal{W} = \{\pm 1\}$.
- Therefore, the potential $f_H =: \mathcal{O}_2 \to \mathbb{C}$ has two singularities, namely $\pm H_0$.

HMS for the adjoint orbit $\mathcal{O}_2$ of $\mathfrak{sl}(2, \mathbb{C})$

Choose in $\mathfrak{sl}(2, \mathbb{C})$ the elements

$$H = H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Hence $\mathcal{O}_2$ is the set of matrices in $\mathfrak{sl}(2, \mathbb{C})$ with eigenvalues ± 1.
- $\mathcal{O}_2$ forms a submanifold of $\mathfrak{sl}(2, \mathbb{C})$ of real dimension 4 (a complex surface).
- In this case the Weyl group is $\mathcal{W} = \{\pm 1\}$.
- Therefore, the potential $f_H =: \mathcal{O}_2 \to \mathbb{C}$ has two singularities, namely $\pm H_0$.

Fukaya–Seidel category for $\mathrm{LG}(2) = (\mathcal{O}_2, f_H)$

Lemma

The Fukaya–Seidel category $\mathit{Fuk}(\mathrm{LG}(2))$ *is generated by two Lagrangians* L_0 *and* L_1 *with morphisms:*

$$\mathrm{Hom}(L_i, L_j) \simeq \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & i < j \\ \mathbb{Z} & i = j \\ 0 & i > j \end{cases} \tag{1}$$

where we think of $\mathbb{Z}$ *as a complex concentrated in degree* 0 *and* $\mathbb{Z}[-1]$ *as its shift, concentrated in degree* 1*, and the products* m_k *all vanish except for* $m_2(\,\cdot\,, \mathrm{I})$ *and* $m_2(\mathrm{I}, \cdot\,)$*.*

Quasi-Vampires

Theorem (Ballico, Barmeier, G. , Grama, San Martin)

- *$LG(2)$ has no projective mirrors.*

Quasi-Vampires

Theorem (Ballico, Barmeier, G. , Grama, San Martin)

- *$LG(2)$ has no projective mirrors.*
- *$\overline{LG(2)}$ has no projective mirrors.*

Quasi-Vampires

Theorem (Ballico, Barmeier, G. , Grama, San Martin)

- *$LG(2)$ has no projective mirrors.*
- *$\overline{LG(2)}$ has no projective mirrors.*

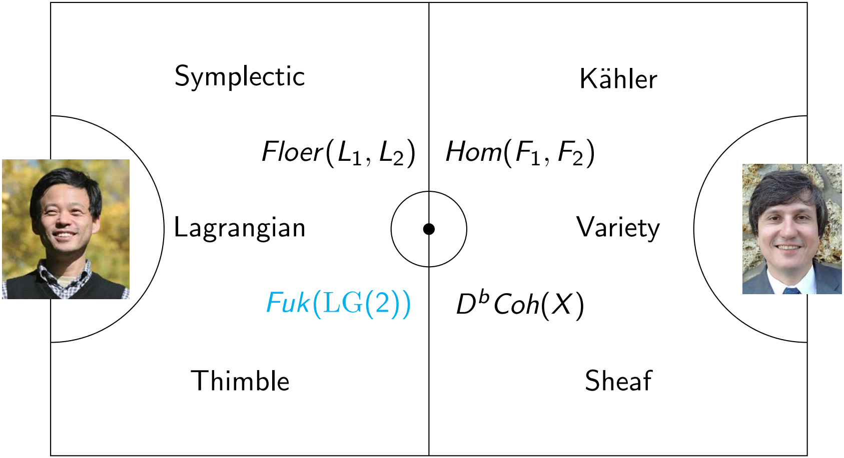
This means:

For any projective variety X we have

$$D^b Coh(X) \neq Fuk(\mathrm{LG}(2))$$

Homological Mirror Match: A1-0B (first half)

A B



Interval coaching

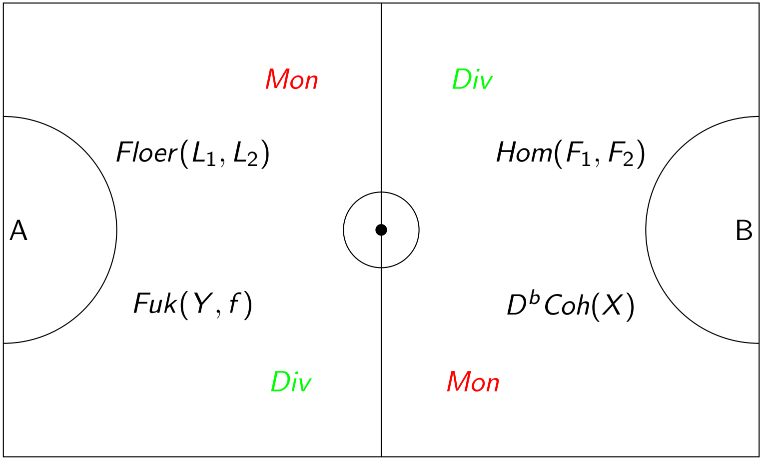
Some Deformation Theory:

The adjoint orbit $Ad(G)H_0$ can be obtained as a deformation of the complex structure of the cotangent bundle of the flag manifold $Ad(K)H_0$.

Example

$T^*\mathbb{P}^1$ deforms to $\mathcal{O}_2$ and $T^*\mathbb{P}^1$ is toric.

Toric Mirror Game (Clarke 2008)



A very symmetric toric LG model

Example

$T^*\mathbb{P}^1$ with the potential $x + y + \dfrac{y^2}{x}$ is mirror to itself.

This self–dual toric LG model deforms to $\mathrm{LG}(2)$ which is not toric.

Homological Mirror Symmetry Conjecture II

For every LG model (Y, f) there exists an LG model (X, g) such that

$$Fuk(Y, f) = D_{Sg}(X, g)$$

where $D_{Sg}(X, g)$ is the Orlov category of singularities.

Orlov's singularity category

$$D_{Sg}(X,g) := \bigoplus_i \frac{D^b Coh(X_i)}{\mathfrak{Perf}(X_i)}$$

where X_i are the critical fibers of g and perfect complexes are (quasi-isomorphic to):

$$0 \to E_1 \to E_2 \to \cdots \to E_n \to 0$$

where E_i are locally free.

Intrinsic Mirror (Gross–Siebert recipe)

- Fit the LG model inside a log Calabi–Yau pair.
- Compute the intersection complex for the dual pair.
- Construct Theta functions.
- Compute punctured Gromov–Witten invariants.

A Landau–Ginzburg mirror for $\mathrm{LG}(2)$

The Landau–Ginzburg model $\mathrm{LG}^\vee(2)$ mirror to $\mathrm{LG}(2)$ obtained using the Gross–Siebert program is:
$\mathrm{LG}^\vee(2) := (X_2, g = y)$ where $X_2 \subset \mathbb{C} \times \mathbb{C}^* \times \mathbb{P}^1$ is given by the equation:

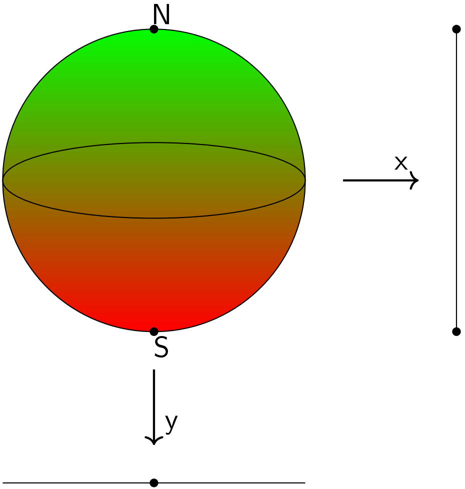
$$uy = v(x + 1 + 1/x).$$

Theorem (Mirror Symmetry for $LG(2)$)

There is an equivalence of categories:

$$D_{Sg}\mathrm{LG}^\vee(2) \simeq Fuk(\mathrm{LG}(2)).$$

Idea



Homological Mirror Match: A1-1B (second half)

A | B

Symplectic | Variety

$Floer(L_1, L_2)$ | $\mathfrak{Perf}$

Lagrangian | Singularity

$Fuk(\mathrm{LG}(2))$ | $D_{Sg}(LG^\vee(2))$

Thimble | Sheaf

The mirror map for $\mathrm{LG}(n)$ with $n > 2$

Theorem (Mirror candidate for $\mathrm{LG}(n)$)

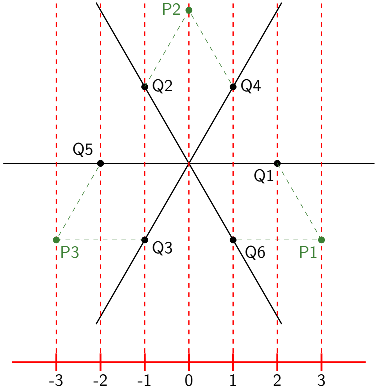
The intrinsic mirror symmetry algorithm produces an LG-model $\mathrm{LG}^\vee(n)$ for which

$$Fuk(\mathrm{LG}(n)) \simeq D_{Sg}\mathrm{LG}^\vee(n)$$

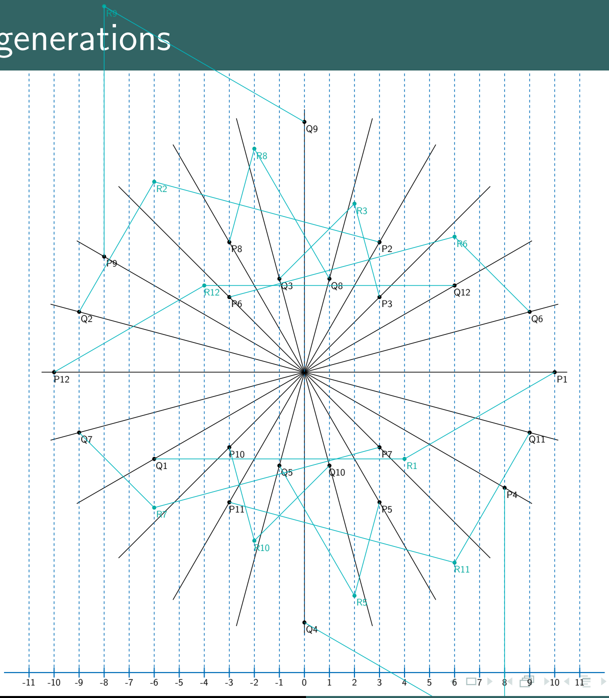
is a 1-1 correspondence.

Degenerations

The sums of critical points is represented in the folllowing picture:



Future degenerations



Definition

For an integers $k > 0$ we set

$$W_k = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1}(k-2)).$$

Bivector fields on $W_1 = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$

The space of all holomorphic bivector fields on W_1 has the following structure as a module over global functions:

$$\langle e_1, e_2, e_3, e_4 \rangle / \langle -zu_2 e_1 + zu_1 e_2 + u_2 e_3 - u_1 e_4 \rangle \ .$$

Here:

$$e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, e_3 = \begin{bmatrix} u_1 \\ z \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} u_2 \\ 0 \\ z \end{bmatrix}$$

For example: $e_3 = u_1 \partial_{u_1} \wedge \partial_{u_2} + z \partial_{u_2} \wedge \partial_z$.

Poisson structures on W_1

The Poisson structures e_1, e_2, e_3, e_4 are all pairwise isomorphic.

W_1 POISSON STRUCTURES

π	degeneracy	Casimir
e_1	$\longrightarrow$	$f(u_2)$

In the case of threefolds, the degeneracy locus is formed by leaves consisting of a single point each, and all other leaves have complex dimension 2.

Bivector fields on $W_2 = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0))$

The space of holomorphic bivector fields on W_2 has the following structure as a module over global holomorphic functions:

$$\langle e_1, e_2, e_3, e_4, e_5 \rangle / \langle u_1 e_3 - z u_1 e_1, u_2 e_5 - z u_2 e_3 - 2 z u_2 e_2 \rangle \ .$$

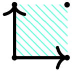
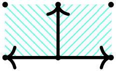
Here:

$$e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} u_1 \\ 0 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ z \\ 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, e_5 = \begin{bmatrix} 2zu_1 \\ z^2 \\ 0 \end{bmatrix} ,$$

and e_1 and e_5 determine isomorphic Poisson structures.

Degeneracy loci on W_2

W_2 Poisson structures

π	degeneracy	Casimir
e_1		$f(u_1)$
e_2		$f(z)$
e_3	 $\cup$ 	$f(u_1)$
e_4	$\emptyset$	$f(u_2)$

Bivector fields on $W_3 = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(+1))$

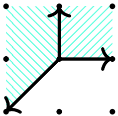
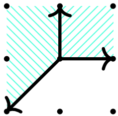
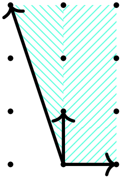
The space of holomorphic bivector fields on W_3 has the following structure as a module over global holomorphic functions:

$$\mathbb{C}\langle e_1, \dots, e_{13}\rangle / R$$

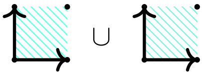
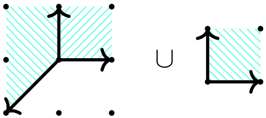
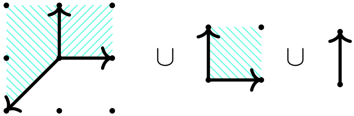
with the set of relations R is the ideal generated by the expressions

$$\begin{array}{lll} u_1 e_2 - u_1 u_2 e_1 & z u_1 e_{13} - u_1 u_2 e_8 & u_1 e_8 - z u_1 e_7 \\ u_1 e_{10} - u_1 u_2 e_3 & u_1 e_{11} - z u_1 e_{10} & u_1 e_6 - z u_1 e_5 - 3z^2 u_1 e_1 \\ u_1 e_{13} - u_1 u_2 e_7 & u_1 e_4 - z u_1 e_3 & u_1 e_9 - z u_1 e_8 + z u_1 e_2 \\ z u_1 e_{12} - u_1 u_2 e_6 & u_1 e_5 - z u_1 e_4 & u_1 e_{12} - z u_1 e_{11} - 3 z u_1 e_1. \end{array}$$

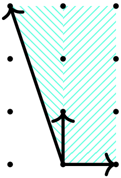
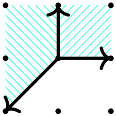
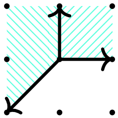
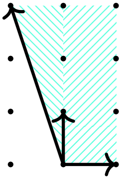
Degeneracy loci on W_3

π	degeneracy	Casimir
e_1	 $\cup$ 	$f(z)$
e_2	 $\cup$ 	$f(z)$
e_3		$f(u_1)$

Degeneracy loci on W_3

π	degeneracy	Casimir
e_4		$f(u_1)$
e_5		$f(u_1)$
e_7		$f(u_2)$
e_8		$f(u_2)$

Degeneracy loci on W_3

π	degeneracy	Casimir
e_{10}	 $\cup$ 	$f(u_1)$
e_{11}	 $\cup$  $\cup$ 	$f(u_1)$
e_{13}	 $\cup$ 	$f(u_2)$

Happy 60th Birthday Michel Van den Bergh